Exercise 11.3

For number fields $K \subseteq L$ and their ring of integer $R \subseteq S$ there is a simpler proof of the fundamental equation,

$$N(PS) = N\Big(\prod_{i=1}^{r} P_i^{e_i}\Big) = \prod_{i=1}^{r} N(P_i)^{e_i} = \prod_{i=1}^{r} [S:P_i]^{e_i}$$

$$= \prod_{i=1}^{r} \Big([R:P]^{f_i}\Big)^{e_i} = N(P)^{\sum_{i=1}^{r} e_i f_i}.$$

Since Cl_R is finite by Thm 8.28 (not true for general Dedekind domains), we have

$$P^h = \lambda R \quad \text{for some } \lambda \in K, \quad h = |Cl_R|$$

Write $\lambda = \frac{a}{b}$, $a, b \in G \Rightarrow bP^h = aR \Rightarrow bP^h S = aS$.

Now, use that N is multiplicative.

$\Rightarrow N(bS)N(P^hS) = N(bP^hS) = N(aS)$

$\Rightarrow N(PS)^h = N(P^hS) = N(aS)N(bS)^{-1} \underset{8.23}{=} |N_{L|\mathbb{Q}}(a)| \cdot |N_{L|\mathbb{Q}}(b)|^{-1} = |N_{L|\mathbb{Q}}(\lambda)|$

Hence,

$$N(PS)^h = |N_{L|\mathbb{Q}}(\lambda)| \overset{\text{Exercise 2.3}}{=} |N_{K|\mathbb{Q}} \circ N_{L|K}(\lambda)|$$

$$\underset{\lambda \in K}{=} |N_{K|\mathbb{Q}}(\lambda)^n| = |N_{K|\mathbb{Q}}(\lambda)|^n = N(\lambda R)^n = N(P^h)^n = N(P)^{hn}$$

$\Rightarrow N(PS) = N(P)^n \Rightarrow \sum_{i=1}^{r} e_i f_i = n.$ $\square$