1) Suppose that P is non-split and unramified (there are only finitely many ramified primes by lecture, so if there are only finitely many nonsplit + unramified, there are only finitely many nonsplit). ①

Let Q be the prime above P.

Since P non-split, $G_Q = G$

Fundamental equation is

$|G| = [L:K] = efr = f$ (using non-split + unramified)

$\Rightarrow |G_Q| = f$

We also have

$$f = [G_Q : I_Q] \Rightarrow I_Q = 1 \Rightarrow G_Q \simeq \mathrm{Gal}_{k(P)}(k(Q))$$

This is a contradiction since $\mathrm{Gal}_{k(P)}(k(Q))$ is cyclic but $G_Q = G$ is not

2) a) Since Q unramified, $I_Q = 1 \Rightarrow G_Q \simeq \mathrm{Gal}_{k(P)}(k(Q))$

$k(P) \subset k(Q)$ is an extension of finite fields

$\Rightarrow \mathrm{Gal}_{k(P)}(k(Q))$ is cyclic and generated by the automorphism

$$\bar{x} \mapsto \bar{x}^q, \quad q = \# k(P), \quad x \in \mathcal{O}_L$$

Thus define σ_Q as the corresponding element in G.

Suppose $\sigma \in G$ is any other automorphism with this property, i.e.

$\sigma(x) \equiv x^q \mod Q$

Claim: $\sigma \in G_Q$.

Proof: Suppose $\sigma \notin G_Q \Rightarrow \sigma Q \neq Q \Rightarrow \exists\, x \in Q$ with $\sigma(x) \notin Q$

$\Rightarrow \sigma(x) \not\equiv 0 \mod Q \Rightarrow \bar{x}^q \not\equiv 0 \mod Q$ ↯ to $x \in Q$.

Uniqueness $\Rightarrow$ G_Q clear now

b) P totally split means $r = n$, $f_i = 1 = e_i$

$\Rightarrow |I_Q| = e = 1$, $[G_Q : I_Q] = f = 1$

$\Rightarrow G_Q$ is trivial $\Rightarrow \left(\frac{L|K}{Q}\right)$ is trivial

Conversely, suppose $\left(\frac{L|K}{Q}\right)$ is trivial.

Since P unramified $\Rightarrow e = 1 \Rightarrow I_Q$ is trivial

$\Rightarrow G_Q \simeq \mathrm{Gal}_{k(P)}(k(Q))$ is trivial

$\Rightarrow f = 1$

$\Rightarrow P$ splits completely.

c) Have $Q' = \tau Q$ for some $\tau \in G$.

$$\sigma_Q(x) \equiv x^{\# k(P)} \mod Q \quad \forall x \in \mathcal{O}_L$$

$$\Rightarrow \quad \tau\, \sigma_Q(x) \equiv \tau(x^{\# k(P)}) \mod \tau Q$$

$$\Rightarrow \quad (\tau \sigma_Q)(x) \equiv \tau(x)^{\# k(P)} \mod Q'$$

This holds for all x, so also for $\tau^{-1}(x)$:

$$(\tau \sigma_Q)(\tau^{-1}(x)) \equiv x^{\# k(P)} \mod Q'$$

$$\Rightarrow \quad \tau \sigma_Q \tau^{-1} = \sigma_{Q'}$$