**Algorithmic Number Theory**
WS 19/20

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# Exercise Sheet 12

Feb 3, 2020

***Exercise 1.*** Show that in a Galois extension of number fields with noncyclic Galois group, there are at most finitely many nonsplit primes and no inert primes.

***Exercise 2.*** Let $K \subseteq L$ be a Galois extension of number fields. Set $G \coloneqq \mathrm{Gal}_K(L)$. Let $P \in \operatorname{Spec} \mathcal{O}_K$ be unramified in $\mathcal{O}_L$.

(a) Let $Q \in \operatorname{Spec} \mathcal{O}_L$ be a prime over $P$. Show that there is a unique element $\sigma_Q \in G$ such that $\sigma(x) = x^{\# k(P)} \bmod Q$ for all $x \in \mathcal{O}_L$.

(b) Instead of $\sigma_Q$ one often writes $\left(\frac{L|K}{Q}\right)$. Show that $P$ is totally split in $\mathcal{O}_L$ if and only if $\left(\frac{L|K}{Q}\right) = 1$ for one (any) $Q \in \operatorname{Spec} \mathcal{O}_L$ above $P$.

(c) Show that $\sigma_Q$ and $\sigma_{Q'}$ are conjugate in $G$ for any $Q, Q' \in \operatorname{Spec} \mathcal{O}_L$ above $P$. Hence, if $G$ is abelian, then $\left(\frac{L|K}{Q}\right)$ is independent of $Q$, and we will simply write $\left(\frac{L|K}{P}\right)$. We thus get a map $\left(\frac{L|K}{\cdot}\right) : \operatorname{Spec}^{ur} \mathcal{O}_K \to G$.