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Übungsblatt 9

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Ring steht hier, wie immer, für einen *kommutativen* Ring (mit 1).

Aufgabe 1. Sei K ein Körper. Wir bestimmen die Normalisierung von

$$A := K[X,Y]/(Y^2 - X^2 - X^3) \ .$$

a) Male ein Bild von A, also von der dazugehörigen Nullstellenmenge.
b) A ist ein Integritätsbereich.[1] Wie deutet das Bild bereits darauf hin?
c) Sei x bzw. y das Bild von X bzw. Y in A. Das Element $\frac{y}{x} \in \mathrm{Q}(A)$ ist ganz über A.
d) Es ist $A \subseteq K[\frac{y}{x}]$.[2]
e) $K[\frac{y}{x}]$ ist die Normalisierung von A.[3]
f) $K[\frac{y}{x}]$ ist ein Polynomring vom Rang 1, die Normalisierung von A ist also die affine Linie über K.[4]

Lösung.
a) *It's a singular cubic curve and looks like this: https://upload.wikimedia.org/wikipedia/commons/a/a6/Cubic_with_double_point.svg*
b) *The picture looks clearly irreducible but we still need to prove it. We claim that the polynomial $p := Y^2 - X^2 - X^3 \in K[X,Y]$ is irreducible. Since $K[X,Y] \simeq (K[X])[Y]$ is a polynomial ring over the factorial ring $K[X]$, it is factorial, so p is a prime element, hence $(p) = (Y^2 - X^3 - X^2)$ is a prime ideal, implying that A is an integral domain.*

The irreducibility of f can be seen as follows (this is not really interesting, perhaps there's a better proof).

Claim: Let A be an integral domain and let $p := Y^2 + \alpha \in A[Y]$ with $\alpha \in A$. Suppose that whenever $\alpha = bd$ in A, then $0 \notin A^\times b + A^\times d$. Then $p \in A[Y]$ is irreducible.

Proof: Suppose that $p = fg$ with $f,g \in A[Y]$ non-zero non-units. Since A is an integral domain, also $A[Y]$ is an integral domain, so we must have $2 = \deg(p) = \deg(f) + \deg(g)$. Without loss of generality, we thus have $\deg(f) = 2$ or $\deg(f) = 1$. If $\deg(f) = 2$, then $f = aY^2 + bY + c$ and $g = d$ with $a,b,c,d \in A$. We must have $ad = 1$, so $a,d \in A^\times$, in particular g is a unit, a contradiction. If $\deg(f) = 1$, then $\deg(g) = 1$, so $f = aY + b$ and $g = cY + d$ with $a,b,c,d \in A$. We have $ac = 1$, so $a,c \in A^\times$. Moreover, $bd = \alpha$ and $ad + bc = 0$. The assumption now implies that this is not possible.

Now, apply this result to $p = Y^2 - \alpha \in (K[X])[Y]$ with $\alpha := -X^3 - X^2$. Suppose that $\alpha = bd$ in $K[X]$. Then without loss of generality $\deg(b) > \deg(d)$. Since $K[X]^\times = K^\times$, we cannot have $0 \in K^\times b + K^\times d$.
c) *Let $p(Z) := Z^2 - x - 1 \in A[Z]$. Then*

$$p\left(\frac{y}{x}\right) = \frac{y^2}{x^2} - x - 1 \ ,$$

[1] Hinweis: Zeige zum Beispiel, dass $Y^2 - X^2 - X^3$ irreduzibel ist. Wieso genügt das?

[2] Hinweis: Kann man vielleicht x und y durch $\frac{y}{x}$ darstellen?

[3] Hinweis: Wir haben eine surjektive Abbildung $K[t] \to K[\frac{y}{x}]$. Also ist $K[\frac{y}{x}]$ Hauptidealring, also?

[4] Hinweis: Betrachte nochmal die Surjektion $\varphi : K[t] \to K[\frac{y}{x}]$. Wir haben dann $K[t]/\operatorname{Ker}\varphi \simeq K[\frac{y}{x}]$. Wir wollen zeigen, dass $P = 0$. Wäre $P \neq 0$, so wäre P maximales Ideal, also....

so
$$x^2 \cdot p\left(\frac{y}{x}\right) = y^2 - x^3 - x^2 = 0 \in A\ .$$
Since A is an integral domain, it follows that
$$p\left(\frac{y}{x}\right) = 0\ ,$$
so $\frac{y}{x}$ is integral over A.

d) *We have $x = \left(\frac{y}{x}\right)^2 - 1$ and $y = \left(\frac{y}{x}\right)^3 - \frac{y}{x}$ which comes directly from the relation $y^2 - x^3 - x^2 = 0$. Since A is as a K-algebra generated by x and y, we thus have $A \subseteq K[\frac{y}{x}]$.*

e) *Since $A \subseteq K[\frac{y}{x}]$ and $K[\frac{y}{x}] \subseteq \mathrm{Q}(A)$, we must have $K[\frac{y}{x}] = \mathrm{Q}(A)$. We clearly have a surjection $K[t] \to K[\frac{y}{x}]$ mapping t to $\frac{y}{x}$. Hence, since $K[t]$ is a principal ideal domain, so is $K[\frac{y}{x}]$. But then $K[\frac{y}{x}]$ is factorial, so integrally closed by a lemma given in the lecture. Since $A \subseteq K[\frac{y}{x}]$ is integral, the integral closure of A in $\mathrm{Q}(A) = \mathrm{Q}(K[\frac{y}{x}])$ is equal to the integral closure of $K[\frac{y}{x}]$ in its fraction field, thus equal to $K[\frac{y}{x}]$.*

f) *Consider again the surjection $\varphi : K[t] \to K[\frac{y}{x}]$ mapping t to $\frac{y}{x}$. This induces an isomorphism $K[t]/\operatorname{Ker}\varphi \simeq K[\frac{y}{x}]$. Since $K[\frac{y}{x}]$ is an integral domain (subring of the field $\mathrm{Q}(A)$), the kernel is a prime ideal, so $\operatorname{Ker}\varphi = P \in \operatorname{Spec}(K[t])$. Since $K[t]$ is a principal ideal domain, we know that either $P = 0$ or P is maximal. Suppose that P is maximal. Then $K[t]/P$, and thus $K[\frac{y}{x}]$ is a field. Since $A \subseteq K[\frac{y}{x}]$ is an integral extension of integral domains, also A is then a field by a lemma in the lecture. But this is clearly a contradiction since $(Y^2 - X^2 - X^3)$ is not a maximal ideal of $K[X, Y]$ since it is contained for example in (X, Y). Hence, we must have $P = 0$, so $K[t] \simeq K[\frac{y}{x}]$.*

Aufgabe 2. Die Eigenschaft von Integritätsbereichen, normal zu sein, ist eine lokale Eigenschaft.

Lösung. *A normal domain is an integral domain which is integrally closed in its field of fractions. Let A be an integral domain. The claim is that A is integrally closed in its field of fractions if and only if A_P is integrally closed in its field of fractions for all $P \in \operatorname{Spec}(A)$.*
First, note that since A is integral, all localizations A_P lie canonically in the fraction field $K := \mathrm{Q}(A)$. In particular, they are integral domains, so it makes sense to talk about normality. Moreover, the fraction field of A_P is also equal to $K := \mathrm{Q}(A)$.
We use the fact that integral closure commutes with localization (proven in the lecture): If $A \subseteq B$ is a ring extension and $S \subseteq A$, then $S^{-1}\mathrm{Int}_A(B) = \mathrm{Int}_{S^{-1}A}(S^{-1}B)$.
Now, suppose that A is normal, so $\mathrm{Int}_A(K) = A$. Then
$$A_P = (\mathrm{Int}_A(K))_P = \mathrm{Int}_{A_P}(K_P) = \mathrm{Int}_{A_P}(K) = \mathrm{Int}_{A_P}(\mathrm{Q}(A_P))\ ,$$
so A_P is integrally closed. Assume conversely that A_P is integrally closed for all $P \in \operatorname{Spec}(A)$. Since $\mathrm{Int}_A(K)$ is a subring of K containing A, it is in particular an A-submodule of K. The same is true for A_P. Now,
$$(\mathrm{Int}_A(K))_P = \mathrm{Int}_{A_P}(K_P) = \mathrm{Int}_{A_P}(K) = \mathrm{Int}_{A_P}(\mathrm{Q}(A_P)) = A_P\ .$$
So, the submodules A and $\mathrm{Int}_A(K)$ coincide locally, hence they are equal (general lemma proven in lecture). This shows that A is integrally closed in K.

Aufgabe 3. Sei $A \subseteq B$ eine ganze Ringerweiterung. Liegt $Q \in \operatorname{Spec}(B)$ über $P \in \operatorname{Spec}(A)$, so ist Q maximal genau dann, wenn P maximal ist.

Lösung. *This is simply going up and incomparability of fibers. First, let P be maximal. Suppose that Q is not maximal. Then there is $Q' \in \operatorname{Spec}(B)$ with $Q \subsetneq Q'$. But then $P = A \cap Q \subseteq A \cap Q'$. Since $A \cap Q' \in \operatorname{Spec}(A)$ and P is maximal, we must have $A \cap Q' = P$. But then Q and Q' are two comparable primes lying over P, contradicting incomparability.*

Conversely, let Q be maximal and suppose that P is not maximal. Then there is $P' \in \operatorname{Spec}(B)$ with $P \subsetneq P'$. By going up there is $Q' \in \operatorname{Spec}(B)$ with $Q \subseteq Q'$ and $Q' \cap A = P'$. Since Q is maximal, we must have $Q = Q'$. Bu then $P = Q \cap A = Q' \cap A = P'$, a contradiction.

Aufgabe 4. Sei $\varphi : A \to B$ ein *ganzer Ringmorphismus*, d.h. $\varphi(A) \subseteq B$ ist ganz. Dann ist $\varphi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *abgeschlossen*, d.h. φ^* bildet abgeschlossene Mengen auf abgeschlossene Mengen ab.

Lösung. *Let us first assume that φ is injective, so $A \subseteq B$ is an integral extension with embedding φ. Note that $\varphi^{-1}(J) = J \cap A$ for any ideal J of B since φ is just a ring extension. In the lecture I've proven that $\varphi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective (lying over). The closed subsets of $\operatorname{Spec}(B)$ are of the form $\mathrm{V}(J) = \{Q \in \operatorname{Spec})(B) \mid Q \supseteq J\}$ for J an ideal of B. I claim that $\varphi^*(\mathrm{V}(J)) = \mathrm{V}(\varphi^{-1}(J)) = \mathrm{V}(J \cap A)$. This implies of course that φ^* is closed.*
Let $Q \in \mathrm{V}(J)$, so $Q \supseteq J$. Then $\varphi^*(Q) = \varphi^{-1}(Q) \supseteq \varphi^{-1}(J)$, so $\varphi^*(Q) \in \mathrm{V}(\varphi^{-1}(J))$.
Conversely, let $P \in \mathrm{V}(\varphi^{-1}(J))$, so $P \supseteq \varphi^{-1}(J)$. Since $A \subseteq B$ is integral, the extension $A/(A \cap J) \subseteq B/J$ is also integral. Since $P \supseteq A \cap J$, we have $P \in \operatorname{Spec}(A/(A \cap J))$. By lying over of the extension $A/(A \cap J) \subseteq B/J$, there is $Q \in \operatorname{Spec}(B)$ with $Q \supseteq J$ lying over P. Hence, $Q \in \mathrm{V}(J)$ and $\varphi^*(Q) = P$, so $P \in \varphi^*(\mathrm{V}(J))$.
This proves the assertion in case φ is injective. For the general case note that we can factorize φ as

$$\begin{tikzcd} A \arrow[r, two heads, "\varphi_0"] \arrow[rr, bend right, "\varphi"'] & \operatorname{Im}\varphi \arrow[r, hook, "\varphi_1"] & B \end{tikzcd}$$

Since Spec is a functor, we have $\varphi^* = (\varphi_1 \circ \varphi_0)^* = \varphi_0^* \circ \varphi_1^*$. By the first part we know that φ_1^* is closed. Since φ_0 is surjective, φ_0^* is an isomorphism between $\operatorname{Spec}(\operatorname{Im}(\varphi_0))$ and the closed subset $\mathrm{V}(\operatorname{Ker}\varphi_0)$. In particular, φ_0^* is closed. The composition of closed maps is closed, hence φ^* is closed.

Aufgabe 5.
Wir zeigen, dass endliche Morphismen endliche Fasern haben.
a) Sei A ein Integritätsbereich, der ein endlich-dimensionaler Vektorraum über einem Körper ist. Dann ist A bereits ein Körper.[5]
b) Ist A eine endlich-dimensionale Algebra über einem Körper, so ist $\operatorname{Spec}(A)$ endlich und besteht nur aus maximalen Idealen.[6]
c) Ist $\varphi : A \to B$ ein endlicher Ringmorphismus (d.h. B ist endlich erzeugter A-Modul), so sind die Fasern von $\varphi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ endlich.

Lösung.
a) *Let K be the base field. Let $0 \neq a \in A$. The multiplication map $f : A \to A$, $x \mapsto ax$, is a K-vector space endomorphism of A. It is injective since A is an integral domain. Hence, it is already surjective, so there is $b \in A$ with $ab = 1$, i.e., a is a unit.*
b) *Let K be the base field and let $n := \dim_K A$. Let $P \in \operatorname{Spec}(A)$. Then A/P is a finite-dimensional K-algebra and it is an integral domain. Hence, A/P is a field by the first part, so P is already maximal. Let $M_1, \ldots, M_r$ be distinct maximal ideals of A. Clearly, $M_i + M_j = A$ for all $i \neq j$, so these are pairwise coprime. The Chinese remainder theorem thus yields a surjective K-algebra morphism $A \twoheadrightarrow A/M_1 \times \ldots A/M_r$, hence $n \geq \dim_K(A/M_1) + \ldots + \dim_K(A/M_r) \geq 1 + \ldots + 1 = r$. So, r is bounded by n.*
c) *The set-theoretic fiber $(\varphi^*)^{-1}(P)$ is in bijection with the underlying set of $\operatorname{Spec}(\mathrm{k}(P) \otimes_A B)$, where $\mathrm{k}(P) := \operatorname{Frac}(A/P)$ is the residue field in P (proven in lecture). If $A \subseteq B$ is finitely generated, so is the extension $\mathrm{k}(P) = \mathrm{k}(P) \otimes_A A \subseteq \mathrm{k}(P) \otimes_A B$, i.e., $\mathrm{k}(P) \otimes_A B$ is a finite-dimensional algebra over a field. Hence, its spectrum is finite by the second part.*

[5] Hinweis: Multiplikation mit einem Element aus A gibt einen Vektorraum-Endomorphismus $A \to A$.

[6] Hinweis: Dass alle Primideale maximal sind, folgt aus dem ersten Teil. Für die Endlichkeit: Haben wir r maximale Ideale, so gilt $\dim_K A \geq r$ mit chinesischem Restsatz!

Aufgabe 6 (Zusatzaufgabe ohne Besprechung[7]). Sei $d \in \mathbb{Z}$ quadratfrei und $L := \mathbb{Q}(\sqrt{d})$. Sei $\mathcal{O}_L := \mathrm{Int}_{\mathbb{Z}}(L)$ die Normalisierung von $\mathbb{Z}$ in L.

a) Es ist

$$\mathcal{O}_L = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{falls } d \equiv 2, 3 \bmod 4\ , \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{falls } d \equiv 1 \bmod 4\ . \end{cases}$$

b) Was ist mit dem Fall $d \equiv 1 \bmod 4$?

c) $\mathcal{O}_L$ ist freier $\mathbb{Z}$-Modul. Bestimme eine $\mathbb{Z}$-Basis.

d) $\mathbb{Z}[\sqrt{-5}]$ is nicht faktoriell. Es gibt also normale Bereiche, die nicht faktoriell sind. [8][9]

Lösung. *This is just a fun exercise to explicitly compute a normalization and to see an example of a normal but not factorial domain. In the book of Eisenbud p.138 there are some hints.*

[7] Hinweise sind im Buch von Eisenbud, S. 138

[8] Hinweis: Man kan $6 \in \mathbb{Z}[\sqrt{-5}]$ auf zwei Arten zerlegen.

[9] Bemerkung: Die $d < 0$, sodass $\mathcal{O}_L$ faktoriell ist, sind bekannt, nämlich $-1, -2, -3, -7, -11, -19, -43, -67, -163$. Für $d > 0$ sieht es schlechter aus: Eine immer noch **offene** Vermutung von Gauß besagt, dass es unendlich viele $d > 0$ mit $\mathcal{O}_L$ faktoriell gibt.