Hironaka's example of a complete but non-projective variety

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Abstract

These are notes for my talk given in Fall 2007 in Barry Mazur's class "Theory of Schemes". My goal is to explain Hironaka's example of a complete but non-projective variety ([Hir60]). I will present Hironaka's construction as in [Har77] and [Šaf94] but I will give more details and try to explain everything precisely. Since Hironaka's construction involves blow-ups I will give a short survey on this important tool concentrating only on those properties we need to understand Hironaka's construction.

Contents

1. Notations

The category of schemes is denoted by Sch, its comma category over a scheme X is denoted by Sch/X. We fix an algebraically closed field k. A **variety** over k is an integral algebraic scheme over k. The category of varieties over k is the full subcategory of Sch/k consisting of varieties over k and is denoted by Var/k.

2. Blow-ups

The reader who is familiar with blow-ups should directly go to section 2.4 or section 3.
A blow-up transforms a subscheme $Y \hookrightarrow X$ into a morphism $\mathrm{Bl}_Y X \xrightarrow{\pi} X$ having certain nice properties with respect to Y. There are different approaches to define and understand blow-ups and each has its advantages: One can describe blow-ups in an abstract way using a universal property, one can describe them as the closure of the graph of a certain geometrically motivated morphism, or (in nice situations), one can describe blow-ups very concretely with equations. The advantage of the abstract approach is that one can immediately understand the nature of blow-ups. However, as any other construction defined by universal properties, its existence has to be proven. This is one reason why the second (geometric) approach becomes important because it can be shown that it satisfies the universal property and proves in this way proves the existence. The equational approach works if both X and Y are regular and is of course most useful for concrete computations.
In the following sections I will review the abstract and the equational approach. This should provide enough information for understanding Hironaka's construction. Thorough discussions of all this can be found in [EH00], [Šaf94] and [Har77].

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2.1. Abstract context of blow-ups

Blow-ups are a special case of an abstract concept that I simply call *transform*. Suppose we are given a category $\mathcal{P}$ over Sch fibered into closed subschemes satisfying a given property $\mathcal{P}$. I will call such a category $\mathcal{P}$-*subscheme category* and call its object over a scheme X the $\mathcal{P}$-subschemes of X.
Suppose that the property $\mathcal{P}$ is an "interesting" property that is not satisfied by all subschemes. Then we would like to have some kind of universal transform that transforms a given subscheme $Y \hookrightarrow X$ into a $\mathcal{P}$-subscheme preserving all information about $X \setminus Y$, so that the transform only affects Y. The following definition gives one formalization of these ideas.

2.1 Definition. A $\mathcal{P}$-*transform* of a closed subscheme $Y \hookrightarrow X$ is a morphism $\widetilde{X} \xrightarrow{\mathcal{P}_{Y/X}} X$ satisfying the following properties:

(i) $\mathcal{P}_{Y/X}^{-1}(Y) \hookrightarrow \mathcal{P}_{Y/X}^{-1}(X) = \widetilde{X}$ is a $\mathcal{P}$-subscheme of $\widetilde{X}$

(ii) $\mathcal{P}_{Y/X}^{-1}(X \setminus Y) \cong X \setminus Y$

(iii) $\mathcal{P}_{Y/X}$ is universal among morphisms satisfying the first two properties, i.e. if $\widetilde{X}' \xrightarrow{\mathcal{P}'_{Y/X}} X$ is another morphism satisfying 1 and 2 then there exists a unique morphism $\widetilde{X}' \to \widetilde{X}$ making the following diagram commutative

$$\begin{tikzcd} \widetilde{X}' \arrow[rr, dashed, "\exists!"] \arrow[dr, "\mathcal{P}'_{Y/X}"'] & & \widetilde{X} \arrow[dl, "\mathcal{P}_{Y/X}"] \\ & X & \end{tikzcd}$$

The closed subscheme $\mathcal{P}_{Y/X}^{-1}(Y) \hookrightarrow \widetilde{X}$ is called the *exceptional subscheme* of the $\mathcal{P}$-transform $\mathcal{P}_{Y/X}$ because it is the only set where $\mathcal{P}_{Y/X}$ is not necessarily an isomorphism. The universality of a $\mathcal{P}$-transform ensures that it is unique up to isomorphism and therefore we can talk about *the* $\mathcal{P}$-transform of a given closed subscheme $Y \hookrightarrow X$. However, as mentioned above, it is not clear that the $\mathcal{P}$-transform exists for a given closed subscheme. This has to be checked for each given property $\mathcal{P}$.

If we suppose that the $\mathcal{P}$-transform exists for any closed subscheme then it is important to know how it behaves under base change: If $Y \hookrightarrow X$ is a closed subscheme and $X' \xrightarrow{f} X$ is a morphism then we get a closed subscheme $f^{-1}(Y) \hookrightarrow X'$ and the obvious question is if and how the $\mathcal{P}$-transform of $f^{-1}(Y) \hookrightarrow X'$ can be obtained from the $\mathcal{P}$-transform of $Y \hookrightarrow X$. One particular way will be given for blow-ups and therefore I define the following notion.

2.2 Definition. Let $\mathcal{P}$ be a subscheme category for which all $\mathcal{P}$-transforms exist. We say that $\mathcal{P}$-transforms are *normal* if for any closed subscheme $Y \hookrightarrow X$ and any morphism $X' \xrightarrow{f} X$ the $\mathcal{P}$-transform of $f^{-1}(Y) \hookrightarrow X'$ is given as follows

$$\begin{tikzcd} \overline{p_2^{-1}(X' \setminus f^{-1}(Y))} \arrow[d, hook] \arrow[dr, "\mathcal{P}_{f^{-1}(Y)/X'}"] & & \\ \widetilde{X} \times_X X' \arrow[r, "p_2"] \arrow[d] & X' \arrow[d, "f"'] & f^{-1}(Y) \arrow[l, hook'] \arrow[d] \\ \widetilde{X} \arrow[r, "\mathcal{P}_{Y/X}"'] & X & Y \arrow[l, hook'] \end{tikzcd}$$

The following Corollary collects some results on normal $\mathcal{P}$-transforms and shows that these transforms are very powerful and have good properties.

2.3 Corollary. Suppose that $\mathcal{P}$-transforms are normal. Let $Y \hookrightarrow X$ be a closed subscheme with $\mathcal{P}$-transform $\mathcal{P}_{Y/X} : \widetilde{X} \to X$.

(i) The inverse image $\mathcal{P}_{Y/X}^{-1}(X \setminus Y)$ is dense in $\widetilde{X}$.

(ii) If $Z \hookrightarrow X$ is another closed subscheme such that Z is not contained in Y then the $\mathcal{P}$-transform of $Z \hookrightarrow X$ is given by the closure of $\mathcal{P}_{Y/X}^{-1}(Z \setminus (Z \cap Y))$ in $\widetilde{X}$. This preimage is called the *proper transform* of Z with respect to $\mathcal{P}_{Y/X}$. The full preimage $\mathcal{P}_{Y/X}^{-1}(Z) \hookrightarrow \widetilde{X}$ is called the *total transform*.

(iii) If $X' \hookrightarrow X$ is an open subscheme then $\mathcal{P}_{X' \cap Y/X'} \cong \mathcal{P}_{Y/X}^{-1}(X') \to X'$ and this isomorphism is unique.

(iv) If $\{U_\lambda\}$ is an open cover of X and $\widetilde{X} \xrightarrow{\pi} X$ is a morphism such that $\pi^{-1}(U) \cong \mathcal{P}_{U \cap Y/U}$ over X then $\pi \cong \mathcal{P}_{Y/X}$.

□

The fourth statement is very important because it shows that normal $\mathcal{P}$-transforms are determined locally. In particular, we have to prove their existence only for closed subschemes of affine schemes.

2.2. Abstract characterization of blow-ups

Now, I will finally define blow-ups as a particular $\mathcal{P}$-transform. For this, I first have to choose the subscheme category.

2.4 Definition. A subscheme $Y \hookrightarrow X$ is called a *Cartier subscheme* if it is locally the zero locus of a single nonzerodivisor, i.e. for all $x \in X$ there is an affine neighborhood $U = \operatorname{Spec}(A)$ such that $Y \cap U = V(f) \subset U$ for some nonzerodivisor $f \in A$.

2.5 Definition. Let $\mathcal{P}$ be the subscheme category of Cartier subschemes. Then the $\mathcal{P}$-transform of a closed subscheme $Y \hookrightarrow X$ is also called the *blow-up* of X along Y and we write $\mathrm{Bl}_{Y/X} : \mathrm{Bl}_Y X \to X$ for this $\mathcal{P}$-transform.

This immediately explains the name "blow-up": a subscheme $Y \hookrightarrow X$ is replaced by the Cartier subscheme $\mathrm{Bl}_{Y/X}^{-1}(Y) \hookrightarrow \mathrm{Bl}_Y X$ which is by definition a hypersurface, i.e. Y is blown-up to a hypersurface. The exceptional subscheme $\mathrm{Bl}_{Y/X}^{-1}(Y) \subseteq \mathrm{Bl}_Y X$ is then also called the *exceptional divisor*.

2.6 Proposition. Blow-ups exist and are normal. In particular, all statements in Corollary 2.3 hold for blow-ups.

Proof. The existence and normality of blow-ups are proven in [EH00], Propositions IV-18 and IV-21. □

The abstract characterization of blow-ups should provide a good understanding of the context and properties of blow-ups. For the discussion of Hironaka's construction we need the concrete equational characterization of blow-ups. This characterization works only if Y and X are both regular schemes but this will be satisfied for our applications. I will discuss this equational characterization in the next two sections beginning with a review on local systems.

2.3. Local systems

Recall that a morphism $(Y, \mathcal{O}_Y) \xrightarrow{(f,\varphi)} (X, \mathcal{O}_X)$ of schemes induces for every $y \in Y$ a natural morphism $\varphi_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ given by $[U, s] \mapsto [f^{-1}(U), \varphi(U)(s)]$. In this way $\mathcal{O}_{Y,y}$ becomes naturally an $\mathcal{O}_{X,f(y)}$-module. In particular, if $Y \hookrightarrow X$ is a closed subscheme then $\mathcal{O}_{Y,y}$ becomes naturally an $\mathcal{O}_{X,y}$-module for every $y \in Y$. This observation is used in the following definition.

2.7 Definition. Let X be a scheme and $Y \hookrightarrow X$ be a closed subscheme. A *local system of equations* for Y in a point $y \in Y$ consists of germs $f_1 \ldots, f_r \in \mathcal{O}_{X,y}$ such that $\mathfrak{m}_{Y,y} = (f_{1,y} \ldots, f_{r,y})\mathcal{O}_{Y,y}$. If $f_{1,y}, \ldots, f_{r,y}$ form a basis of the $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$-vector space $\mathfrak{m}_{Y,y}/\mathfrak{m}_{Y,y}^2$ then it is called a *local system of parameters* for Y at y.

By Nakayama's Lemma, every local system of parameters is a local system of equations but the converse does not hold in general. To understand the geometric motivation for this definition, one has to remember that the maximal ideal $\mathfrak{m}_{Y,y}$ consists precisely of those regular function on Y which are defined in a neighborhood of y and vanish at this point. The existence of a regular system of equations in y shows that all functions on Y vanishing in y are already determined by functions on X vanishing in y. If this system is moreover a system of parameters, then all these functions additionally vanish with different tangent directions in y.

2.8 Theorem. Let $X \in \mathrm{Var}(k)$ and let $Y \hookrightarrow X$ be a closed subvariety. Then around each closed point $y \in Y$ which is regular in *both* X and Y there exists a local system of parameters of Y in y consisting of $r = \operatorname{codim}_X Y$ functions. If both X and Y are regular then around each point $y \in U$ there exists functions $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $f_{1,x}, \ldots, f_{r,x}$ is a local system of parameters for Y in each $x \in U \cap Y$.

Proof. See [Šaf94], p. 71. □

So, if both X and Y are regular then Y can locally defined by functions on X which vanish on Y with different tangent directions.

2.9 Example.
(i) Let $X \subset \mathbb{A}^2$ be a smooth curve. Then each local ring at a closed point is a discrete valuation ring which admits a uniformizing parameter defining a local system of parameters.
(ii) Let $X = V(T_2(T_2 - T_1^2), T_3) \subset \mathbb{A}^3$. But around the origin $(0,0,0) \in X$ we always have functions vanishing on the line and on the parabola and all these functions vanish with the same tangent direction in the origin. Hence, there is no local system of parameters around the origin.

2.4. Equational characterization of blow-ups

Let $X \in \mathsf{Var}(k)$ be a regular n-dimensional affine variety and let $Y \hookrightarrow X$ be a closed regular subvariety. Since blow-ups are determined locally, it is enough to assume that we have a single local system of parameters $u_1, \dots, u_r$ describing Y in X and $r = \operatorname{codim}_X Y$.
Let $\mathrm{Bl}_Y X$ be the closed subset of $X \times \mathbb{P}^{r-1}$ defined by the equations

$$t_i u_j = t_j u_i \subseteq X \times \mathbb{P}^{r-1} \text{ for } i, j = 1, \dots, r,$$

where the t_i are the homogeneous coordinates of $\mathbb{P}^{r-1}$. The projection from $X \times \mathbb{P}^{r-1}$ onto the first factor then defines a morphism $\mathrm{Bl}_Y X \xrightarrow{\pi} X$.

2.10 Proposition. The morphism $\mathrm{Bl}_Y X \xrightarrow{\pi} X$ is the blow-up of X along Y.

Proof. See [EH00], Exercise IV-26. □

2.11 Proposition. The blowup $\mathrm{Bl}_Y X \xrightarrow{\pi} X$ has the following properties:
(i) $\pi^{-1}(Y) = Y \times \mathbb{P}^{r-1}$ and $\pi^{-1}(y) \cong \mathbb{P}^{r-1}$ for each $y \in Y$.
(ii) $\mathrm{Bl}_Y X \setminus \pi^{-1}(Y) \xrightarrow{\pi} X \setminus Y$ is an isomorphism.
(iii) The construction of $\mathrm{Bl}_Y X$ does not depend on the choice of the local system of parameters for Y.
(iv) $\mathrm{Bl}_Y X$ is irreducible, regular and of dimension n.

Proof. The first statement can be derived from the equations. The second statement is obvious and the third statement is due to the fact that the blow-up is unique up to isomorphism. For the last statement, see [Šaf94]. □

To globalize this construction to blow-ups of a regular variety $X \in \mathsf{Var}(k)$ along a regular closed subvariety Y, we choose an affine cover U_α of Y such that Y is on each U_α defined by a local system of parameters $u_{\alpha,1}, \dots, u_{\alpha,r}$. Then we blow up U_α along $Y \cap U_\alpha$ and get a family of morphisms $\mathrm{Bl}_{Y \cap U_\alpha} \xrightarrow{\pi_\alpha} U_\alpha$. On each intersection $U_\alpha \cap U_\beta$ we have two local systems of parameters and according to the uniqueness of the local blow-up construction from above there exists a unique isomorphism

$$\pi_\alpha^{-1}(U_\alpha \cap U_\beta) \xrightarrow[\cong]{\varphi_{\alpha\beta}} \pi_\beta^{-1}(U_\alpha \cap U_\beta)$$

We can glue the $\mathrm{Bl}_{Y \cap U_\alpha}$ along these isomorphisms and get a variety $\mathrm{Bl}_Y X$ and by gluing the π_α we get a morphism $\mathrm{Bl}_Y X \xrightarrow{\pi} X$. This morphism is then the blow-up of X along Y. Since the local blow-up construction is unique and independent of the local systems of parameters, this construction is also independent of the chosen local systems of parameters.

3. Hironaka's example

Hironaka varieties are examples of non-singular and complete but non-projective varieties. It was proven that 2-dimensional non-singular complete varieties are projective (see [Šaf94]) and therefore Hironaka varieties are important counter examples showing that this is already in dimension 3 not true.

3.1. Hironaka varieties

Hironaka has not given only one counter example but a whole class of counter examples. Therefore I introduced the name *Hironaka variety* for his construction. These varieties are constructed as follows.
Let $X \in \mathsf{Var}(k)$ be a regular projective threefold over k containing two regular rational curves C, D intersecting transversally in two points P, Q. The objects (k, X, C, D, P, Q) are the "variables" in the construction producing the different counter examples. To show that at least one such tuple exists, we can take $k = \mathbb{C}$, $X = \mathbb{P}^3_{\mathbb{C}} = \operatorname{Proj}\mathbb{C}[T_0, T_1, T_2, T_3]$, $C = V(T_2, T_3)$ and $D = V(T_0^2 - T_1^2 + T_0 T_2, T_3)$.

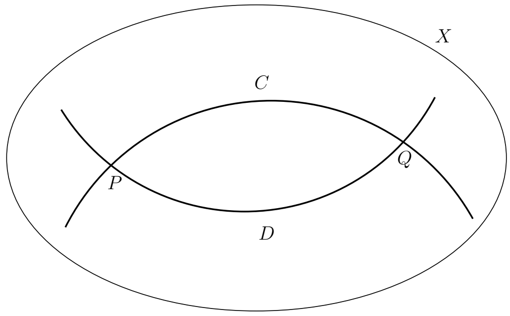

Figure 1: Initial setup

Now, consider the following two sequences of blow-ups

$$X_1 = \mathrm{Bl}_{(D\setminus P)'}\left(\mathrm{Bl}_{C\setminus P}(X \setminus P)\right) \xrightarrow{\pi_2} \mathrm{Bl}_{C\setminus P}(X \setminus P) \xrightarrow{\pi_1} X \setminus P$$
$$X_2 = \mathrm{Bl}_{(C\setminus Q)'}\left(\mathrm{Bl}_{D\setminus Q}(X \setminus Q)\right) \xrightarrow{\sigma_2} \mathrm{Bl}_{D\setminus Q}(X \setminus Q) \xrightarrow{\sigma_1} X \setminus Q,$$

where $(D \setminus P)'$ is the proper transform of $D \setminus P$ in $\mathrm{Bl}_{C\setminus P}(X \setminus P)$ and $(C \setminus Q)'$ is the proper transform of $C \setminus Q$ in $\mathrm{Bl}_{D\setminus Q}(X \setminus Q)$. Let $\pi = \pi_2\pi_1$, $\sigma = \sigma_2\sigma_1$ and let $U = X \setminus \{P, Q\}$ be the curve obtained by removing the two intersection points of C and D from X.
Before looking at the result in detail we first try to figure out what happens to the curve U after blowing up, i.e. we want to compare $\pi^{-1}(U)$ and $\sigma^{-1}(U)$. Note that the curve U is disconnected and therefore, as indicated in figure 2, we can find an affine cover U_α of U in X such that each U_α intersects U in only one of its six components.

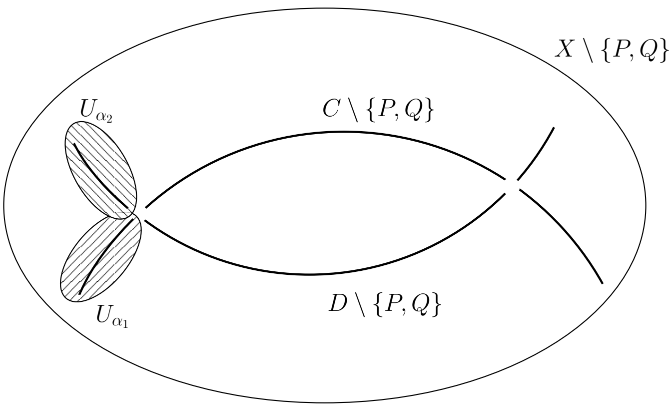

Figure 2: Isolating the components of U

But then, as blow-ups are determined locally and are isomorphisms away from the exceptional divisor, we immediately see that on U it does not matter if we blow up first along C and then along D or first along D and then along C. In other words, because we removed the intersection points of the curves the local blow-ups do not interfere anymore and so we do not have to care about the order. Therefore $\pi^{-1}(U)$ and $\sigma^{-1}(U)$ are isomorphic and we can glue X_1 and X_2 along this isomorphism and obtain a variety H. By gluing π and σ we also get a morphism $H \xrightarrow{f} X$. This morphism is called the *Hironaka variety* obtained from the data (k, X, C, D, P, Q). The construction is illustrated on the next page.

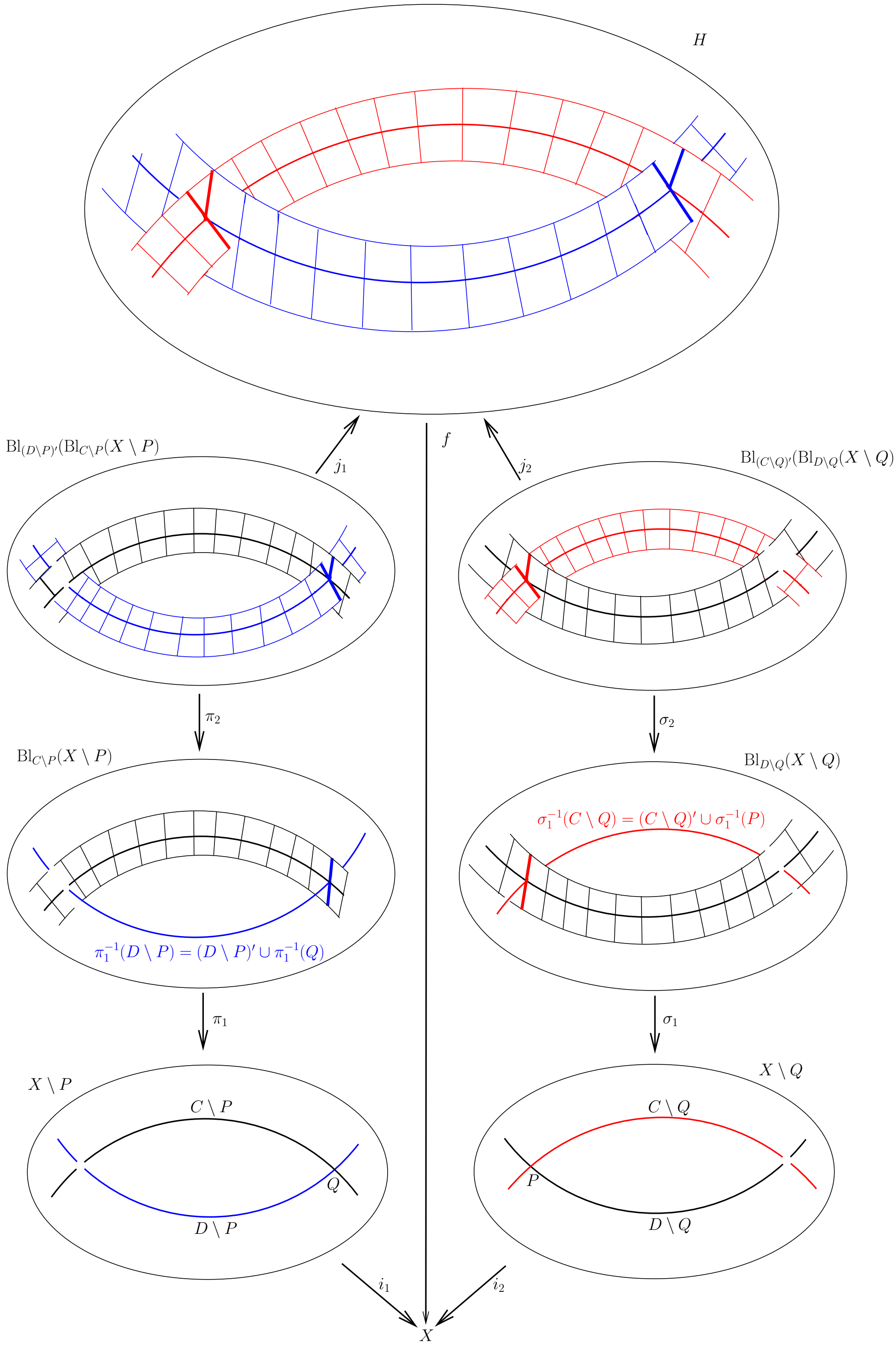

Figure 3: Construction of a Hironaka variety
(note that I have used some so far unproven but "geometrically obvious" assumptions in the pictures, see section 3.3)

3.2. Some notations and observations concerning Hironaka varieties

By the properties of our initial setup and by the properties of blow-ups, H is a regular variety over k. We denote the exceptional hypersurface of the blow-up π_1 resp. σ_1 by E_1 resp. F_1. In the picture, this is the black surface containing one blue resp. red line in the first step. The exceptional hypersurfaces of the blow-up π_2 resp. σ_2 is denoted by E_2 resp. F_2. In the picture, this is the blue resp. red surface in the second step.
Our gluing process glues the surfaces E_1 and F_2 resp. E_2 and F_1 to a hypersurface S_1 resp. S_2 in H. In the picture, this is the red resp. blue surface in H. We denote the hypersurface $S_1 \cup S_2$ by S. We then have morphisms $f_1 := f|_{S_1} : S_1 \to C$ and $f_2 := f|_{S_2} : S_2 \to D$.

3.3. Non-projectivity of Hironaka varieties

The proof of the non-projectivity of a Hironaka variety is based on a deduction of a statement in its intersection theory which cannot hold for projective varieties. In detail, we construct Weil divisors on the surface S which have positive degree but add up to zero. This would not be possible if the variety is projective. The key fact for deriving this contradiction is more or less simply the statement that figure 3 is correct (this is the unproven assumption I was talking about). The following Proposition makes precise what I mean.

3.1 Proposition. Let $H \xrightarrow{f} X$ be a Hironaka variety with data as above. Then $f^{-1}(P)$ decomposes into two distinct projective lines L_Q and L'_Q both contained in S_1. The same holds for $f^{-1}(Q)$.

We prove this Proposition in two steps. First we prove a statement that I have also assumed in drawing the first step of the picture.

3.2 Lemma.

1. The total transform $\pi_1^{-1}(D \setminus P)$ of $D \setminus P$ decomposes into two irreducible components
$$\pi_1^{-1}(D \setminus P) = (D \setminus P)' \cup \pi_1^{-1}(Q),$$
where $(D \setminus P)'$ is the proper transform of $D \setminus P$.
2. $(D \setminus P)'$ is isomorphic to the curve $D \setminus P$ and $\pi_1^{-1}(Q)$ is a projective line.
3. The two components intersect transversally in exactly one point Q'.

Proof.
1. This is obvious since $(D \setminus P)' = \overline{\pi_1^{-1}(D \setminus \{P, Q\})}$. The irreducibility can be derived from the next statement.
2. That $\pi_1^{-1}(Q)$ is a projective line follows simply from the properties of blow-ups. That $(D \setminus P)'$ is isomorphic to $D \setminus P$ is also obvious since $(D \setminus P)'$ is the closure of $\overline{\pi_1^{-1}(D \setminus \{P, Q\})}$ in the blow-up. But $\pi_1^{-1}(D \setminus \{P, Q\})$ is isomorphic to $D \setminus \{P, Q\}$ and the closure just "adds the one missing point on the curve".
3. We can restrict to an affine neighborhood around $\pi^{-1}(Q)$ and as the two curves C, D intersect transversally we can assume the following situation: $X = \mathbb{A}^3$, $C = \{x = y = 0\}$ is the z-axis, $D = \{y = z = 0\}$ is the x-axis and Q is the origin. Let $\mathrm{Bl}_Y X \xrightarrow{\pi_1} X$ be the blow-up of X along C. We want to show that the proper transform of D in $\mathrm{Bl}_Y X$ intersects $E = \pi_1^{-1}(C)$ transversally in one point.
The blow-up $\mathrm{Bl}_Y X \subset \mathbb{A}^3 \times \mathbb{P}^1$ is given by the equation $t_1 y = t_2 x$, where t_1, t_2 are the homogeneous coordinates on $\mathbb{P}^1$. We can cover the blow-up $\mathrm{Bl}_Y X \subset \mathbb{A}^3 \times \mathbb{P}^1$ with two coordinate charts $U_1 = \{t_1 \neq 0\}$ and $U_2 = \{t_2 \neq 0\}$. On U_1, we can set $t_1 = 1$ and $\mathrm{Bl}_C X \cap U_1$ is then given by $y = t_2 x$. Similarly, $\mathrm{Bl}_C X \cap U_2$ is given by $x = t_1 y$. The total transform of D in $\mathrm{Bl}_C X$ given by the equation $y = z = 0$ and on the open subsets U_1,U_2 we get

$$\begin{aligned}
\mathrm{Bl}_C X \cap U_1 \cap \pi^{-1}(D) &= \{y = t_2 x, y = z = 0\} = \{t_2 x = y = z = 0\} \\
&= \{x = y = z = 0\} \cup \{y = z = t_2 = 0, x \neq 0\} = D' \\
\mathrm{Bl}_C X \cap U_2 \cap \pi^{-1}(D) &= \{x = t_1 y, y = z = 0\} = \{x = y = z = 0\} = \pi^{-1}(Q)
\end{aligned}$$

We see that the total transform of D decomposes into the line l_Q and into the proper transform $D' \cong D$. Moreover, both components intersect transversally in the origin. □

Similar statements hold of course "on the opposite side", i.e for Q replaced by P and π_1 replaced by σ_1.

Proof of Proposition 3.1. Obviously, $f^{-1}(Q)$ decomposes into the two projective lines $L'_Q = \pi_2^{-1}(Q')$ and $L_Q = \pi_1^{-1}(Q)'$, where Q' is the unique intersection point from Lemma 3.2 and $\pi_1^{-1}(Q)'$ is the proper transform of $\pi_1^{-1}(Q)'$ under the blow-up π_2. The line L_Q is obviously contained in S_1.
To show that L'_Q is contained in S_1, one can use the Lemma above and a similar[1] simplified situation as in its proof. $\square$

Now, let $H \xrightarrow{f} X$ be a Hironaka variety obtained by the data (k,X,C,D,P,Q). Choose two more points $A \in C \setminus \{P, Q\}$ and $B \in D \setminus \{P, Q\}$. Since the curves C, D are rational curves, any two points on them are linearly equivalent. Linear equivalence is preserved under pullbacks of divisors and therefore we get the following linear equivalences (we keep track of the variety where the equivalence holds using an index at the tilde):

$$A \sim_C Q \Longrightarrow f_1^{-1}(A) \sim_{S_1} f_1^{-1}(Q) = L_Q + L'_Q$$
$$B \sim_D P \Longrightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(P) = L_P + L'_P.$$

Using the pushforwards of cycles under the inclusions $S_1 \overset{j_1}{\hookrightarrow} S$ and $S_2 \overset{j_2}{\hookrightarrow} S$, we get equivalences on S:

$$I : f^{-1}(A) \sim_S f^{-1}(Q) = L_Q + L'_Q$$
$$II : f^{-1}(B) \sim_S f^{-1}(P) = L_P + L'_P.$$

B and Q lie on D and therefore, as D is rational, we have

$$III : B \sim_D Q \Rightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(Q) \Rightarrow f^{-1}(B) \sim_S L'_Q.$$

Similarly we get

$$IV : f^{-1}(A) \sim_S L'_P.$$

Combining all these equivalences, we get the following equivalence

$$\begin{aligned} f^{-1}(A) + f^{-1}(B) \sim_S f^{-1}(A) + f^{-1}(B) &\Rightarrow L_Q + L'_Q + L_P + L'_P \sim_S L'_Q + L'_P \\ &\Rightarrow L_Q + L_P \sim_S 0. \end{aligned}$$

Now, suppose that H would be projective. Then the both lines L_Q and L_P would have a positive degree. The degree is a $\mathbb{Z}$-linear map on the cycles and therefore the sum L_Q and L_P would also have a positive degree. But on a projective variety the trivial cycle has degree 0 contradicting the linear equivalence above.

3.4. Completeness of Hironaka varieties

It remains to show that Hironaka varieties are complete. So, let $H \xrightarrow{f} X$ be a Hironaka variety as above. We have to show that for any variety Z the projection $H \times Z \xrightarrow{p} Z$ is closed. The projection p can be factored as $p = q \circ (f \times \mathrm{id})$ where $X \times Z \xrightarrow{q} Z$ is the projection. Since X is projective by assumption, q is a closed map and therefore it suffices to show that $f \times \mathrm{id}$ is closed. X can be covered by the two open sets $U_1 = X \setminus P$ and $U_2 = X \setminus Q$ and as closed is a local property, it suffices to show that the restrictions

$$f \times \mathrm{id} : (f \times \mathrm{id})^{-1}(U_i \times Z) \to U_i \times Z$$

are closed. The map f is over U_i a composition of blow-ups and it suffices to show that blow-ups are closed. Since blow-ups are local, we can check this on an affine open on which our blow-up is defined by the local construction. But such a blow-up is simply a projection $U \times \mathbb{P}^{r-1} \to U$ which is closed as $\mathbb{P}^{r-1}$ is complete.

References

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, 1977.

[Hir60] Heisuke Hironaka, *On the theory of birational blowing-up*, Thesis (unpublished), 1960.

[Šaf94] Igor' R. Šafarevič, *Basic algebraic geometry. 2*, second ed., Springer-Verlag, Berlin, 1994, Translated from the 1988 Russian edition by Miles Reid.

[1] Unfortunately, I couldn't work this out, so I hope it works similarly. I give this as an exercise.