COMMENTS ON THE BOOK "TENSOR CATEGORIES" BY ETINGOF, GELAKI, NIKSHYCH & OSTRIK

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This document contains some comments on the book "Tensor Categories" by Etingof, Gelaki, Nikshych & Ostrik [3]. I've written them while reading the book and giving seminars and courses based on it. They consist of minor—sometimes trivial—remarks, corrections or additions, issues I couldn't resolve, and things I found interesting. If you find a mistake or can resolve one of the issues, please send me an email. Note that P. Etingof published some corrections on his website[1] as well. I want to emphasize that I consider this an important and excellent book, even though some of my comments may sound a bit sloppy.

I would like to thank Liam Rogel for turning my comments—which I originally had on my website—into this LaTeX document.

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1. Chapter 1: Abelian categories

1.1. Footnote page 1.

> In fact, even when we work with categories that are not essentially small (such as the category of all vector spaces), we will allow ourselves to abuse terminology and speak about "the set of isomorphism classes of objects" of such a category.

So, for example in Section 1.5 where the assumption is only that $\mathcal{C}$ is abelian we need to implicitly assume that it is also essentially small since otherwise the Grothendieck

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[1] http://www-math.mit.edu/~etingof/tensbookcor.pdf

group can't be defined. I guess this is what the footnote says. Such an implicit assumption seems to be in place throughout the book whenever necessary I think.

1.2. **Section 1.10.** The coend of a functor $F\colon \mathcal{C} \to \mathrm{Vec}$ is a quotient of $\bigoplus_{X \in \mathcal{C}} F(X)^* \otimes F(X)$. This thing needs to be a vector space, so I would say $\mathcal{C}$ in fact needs to be small. The statements in Theorem 1.9.15 and Theorem 1.10.1 are something up to equivalence, so essentially small is enough here (this is added in Theorem 1.9.15 but not in Theorem 1.10.1). [Update: I was told that the coend can still be defined in the essentially small case by summing over representatives of the isomorphism classes of simple objects.]

2. Chapter 2: Monoidal categories

2.1. **Historical remark.** Category theory was introduced by S. Eilenberg and S. MacLane [2] in 1945. The definition of a monoidal category first appeared in the paper [6] by S. MacLane (1963). From the same year there's also the paper [1] by J. Bénabou, in which a category with multiplication is defined. I do not have access to the last paper but judging from the MathSciNet review (and from the fact that the paper is only 3 pages long), coherence in not discussed here. M. Müger says "It is mysterious to this author why the explicit formalization of tensor categories took twenty years to arrive after that of categories [...]". I'm not sure if I agree.

2.2. **Alternative structure.** I think it's nicer to rearrange the material a bit and proceed more like in MacLane's 1963 paper: introduce categories with multiplication; introduce associators as the first stage in categorifying associativity; say that we also want "higher associativity laws" (without formally defining what this means since it's "intuitive"); for n factors there are $C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ possible ways to set the parenthesis; for semigroups we get higher associativity for free; for categories, higher associativity for $n = 4$ is precisely the pentagon diagram ($C_3 = 5$); MacLane's coherence theorem says that once we impose higher associativity for $n = 4$ we have it for all n. I think one can believe that without a proof. A category with multiplication and associator satisfying higher associativity (so, satisfying the pentagon axiom) is called a semigroup category. This categorifies the notion of a semigroup.

Remark 2.1. The proof of the coherence theorem given in the book (Theorem 2.9.2) uses the strictness theorem (Theorem 2.8.5) and, as far as I can see, the proof of this uses the unit to show that the functor $L\colon \mathcal{C} \to \mathcal{C}^l$ is fully faithful. I don't know if one can show this directly without a unit. But the proof of the coherence theorem in MacLane's paper works without a unit, so all is good. (In Remark 2.2.9 it is stated that semigroup categories categorify semigroups. However, as the proof of the coherence theorem given there implicitly seems to make use of the unit, I would say this is not clear from there.)

2.3. **Second paragraph of Section 2.1.** "Abelian categories are a categorification of abelian groups". Is this not a bit strong?

2.4. **Motivation Units.** To introduce units, I again think it's nicer to rearrange things.

A unit in a semigroup is an element 1 such that $1 \cdot x = x$ and $x \cdot 1 = x$ for all x. The first stage in categorifying this is a triple $(1, l, r)$ of an object $1 \in \mathcal{C}$ and natural equivalences $l\colon 1 \otimes - \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ and $r\colon -\otimes 1 \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$. But, similar as before with the higher associativity, we want to freely insert and remove the 1 everywhere. For semigroups we get this for free; but for semigroup categories we need to impose it. Considering three factors we want for example that the diagrams

$$\begin{tikzcd}(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}(1 \otimes X) \otimes Y \arrow[rr, "a_{1,X,Y}"] \arrow[dr, "l_X \otimes \mathrm{id}_Y"'] & & 1 \otimes (X \otimes Y) \arrow[dl, "l_{X \otimes Y}"] \\ & X \otimes Y & \end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd}(X \otimes Y) \otimes 1 \arrow[rr, "a_{X,Y,1}"] \arrow[dr, "r_{X \otimes Y}"'] & & X \otimes (Y \otimes 1) \arrow[dl, "\mathrm{id}_X \otimes r_Y"] \\ & X \otimes Y & \end{tikzcd} \tag{2.3}$$

commute and that

$$l_1 = r_1\ .$$

Let's call such a structure an LR unit. The “extended” coherence theorem (Theorem 2.9.2, or the one in MacLane's paper) shows that these conditions already imply coherence including the unit object for arbitrary number of factors.

LR units were introduced by MacLane (1963). In the paper [4] by G.M. Kelley (1964), it is shown that the first diagram above (called triangle diagram) already implies the other three properties. This is also shown in the book in Proposition 2.2.4 and Corollary 2.2.5.

It is straightforward to see that a unit in a semigroup can equivalently be characterized by the property that $1^2 = 1$ and the maps $x \mapsto 1 \cdot x$ and $x \mapsto x \cdot 1$ are bijections. This point of view was categorified by N. Saavedra Rivano [7] in 1972, under the name reduced units. A reduced unit is a pair $(1, \iota)$ of an object $1 \in \mathcal{C}$ and an isomorphism $1 \otimes 1 \xrightarrow{\sim} 1$ such that the functors $L_1\colon \mathcal{C} \to \mathcal{C}$ defined by $X \mapsto 1 \otimes X$, $f \mapsto \mathrm{id}_1 \otimes f$, and $R_1\colon \mathcal{C} \to \mathcal{C}$ defined by $X \mapsto X \otimes 1$, $f \mapsto f \otimes \mathrm{id}_1$, are equivalences on $\mathcal{C}$.

In the obvious way one can define a category of LR units and a category of reduced units. The arguments in the book show essentially:

Proposition 2.2. *The category of LR units is canonically isomorphic to the category of reduced units. Moreover, these categories are contractible if not empty, i.e., they* *are equivalent to the terminal category (in particular up to unique isomorphism, there's just one object).*

A semigroup category having a unit is called a monoidal category. The approach using reduced units shows that a unit for a semigroup category is a property and not a structure as there's no condition on ι. Reduced units are a more economical way to encode the same information.

Remark 2.3. Saavedra (1972) showed that both notions of units are equivalent, but there seems to be a mistake in the proof given there according to the nice paper J. Kock [5]. The proposition above is also in Kock's paper.

2.5. **Beginning of 2.2.** Why do l_X and r_X exist? Recall that $L_1 = 1 \otimes - \colon \mathcal{C} \to \mathcal{C}$ is an equivalence. In particular, it induces an isomorphism

$$(L_1)_{1\otimes X,X} \colon \mathrm{Hom}(1 \otimes X, X) \to \mathrm{Hom}(1 \otimes (1 \otimes X), 1 \otimes X) \ . \tag{2.4}$$

We have a morphism

$$1 \otimes (1 \otimes X) \xrightarrow{a^{-1}_{1,1,X}} (1 \otimes 1) \otimes X \xrightarrow{\iota \otimes \mathrm{id}_X} 1 \otimes X \ , \tag{2.5}$$

and now we define l_X to be the inverse of this one under $(L_1)_{1\otimes X,X}$. We proceed similarly for r_X.

2.6. **Proof of Proposition 2.2.2.** The commutative diagram comes from naturality of $l \colon 1 \otimes - \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ applied to $l_X \colon 1 \otimes X \to X$.

2.7. **Proof of Proposition 2.2.3.** Why do the quadrangles commute? As stated, it's due to the functionality of the associativity isomorphisms. But maybe it's helpful unraveling this once because arguments like this will be used all the time. Let's look at the left quadrangle and write it like this:

$$\begin{tikzcd} ((X \otimes 1) \otimes 1) \otimes Y \arrow[r, "r_X \otimes \mathrm{id}_1 \otimes \mathrm{id}_Y"] \arrow[d, "a_{X\otimes 1,1,Y}"'] & (X \otimes 1) \otimes Y \arrow[d, "a_{X,1,Y}"] \\ (X \otimes 1) \otimes (1 \otimes Y) \arrow[r, "r_X \otimes \mathrm{id}_{1\otimes Y}"'] & X \otimes (1 \otimes Y) \end{tikzcd} \tag{2.6}$$

Recall that if you have functors $F, G \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi \colon F \to G$, then naturality of φ means that whenever you have a morphism $f \colon X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\varphi_X"'] & F(Y) \arrow[d, "\varphi_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd} \tag{2.7}$$

commutes. One needs to stare a minute at the diagram before to see what the F, G, X, Y, φ, f are that makes it a naturality diagram. They are

$$F = (-\otimes 1)\otimes Y \ , \quad G = -\otimes(1\otimes Y) \ , \quad \varphi = a_{-,1,Y} \colon F \to G \ , \quad f = r_X \colon X\otimes 1 \to X \ . \tag{2.8}$$

2.8. **Corollary 2.2.5.** Typo at the beginning, set $X = Y = 1$ in (2.12).

2.9. **Proposition 2.2.6.** In the proof it's not really said what's meant by “η” maps ι to “ι'”. It means that the diagram

$$\begin{tikzcd} 1 \otimes 1 \arrow[r, "\eta\otimes\eta"] \arrow[d, "\iota"'] & 1' \otimes 1' \arrow[d, "\iota'"] \\ 1 \arrow[r, "\eta"'] & 1' \end{tikzcd} \tag{2.9}$$

commutes. This is exactly what a morphism in the category of reduced units is. To prove this (Exercise 2.2.7), proceed as follows. We have a commutative diagram

$$\begin{tikzcd} & 1 \otimes Y \arrow[rr, "l_Y"] & & Y \\ (1 \otimes 1') \otimes Y \arrow[ur, "r'_1\otimes\mathrm{id}_1"] \arrow[rr, "a_{1,1',Y}"] \arrow[dr, "l_{1'}\otimes\mathrm{id}_Y"'] & & 1 \otimes (1' \otimes Y) \arrow[ul, "\mathrm{id}_1\otimes l'_Y"'] \arrow[r, "l_{1'\otimes Y}"] \arrow[dl, "l_{1'\otimes Y}"] & 1' \otimes Y \arrow[u, "l'_Y"] \arrow[d, "l'_Y"] \\ & 1' \otimes Y \arrow[rr, "l_{Y'}"'] & & Y \end{tikzcd} \tag{2.10}$$

The upper triangle is the triangle diagram, the lower triangle is the additional triangle diagram (2.12) from Proposition 2.2.4 applied to $X = 1'$ and $Y = Y$. The upper square is naturality and the lower square is obvious. The outer path gives the commutative diagram

$$\begin{tikzcd} 1 \otimes Y \arrow[dr, "l_Y"] \arrow[dd, "\eta\otimes\mathrm{id}_Y"'] & \\ & Y \\ 1' \otimes Y \arrow[ur, "l'_Y"'] & \end{tikzcd} \tag{2.11}$$

In a similar way one obtains, using the second triangle diagram (2.13) from Proposition 2.2.4, the diagram

$$\begin{tikzcd} X \otimes 1 \arrow[dr, "r_X"] \arrow[dd, "\mathrm{id}_X\otimes\eta"'] & \\ & X \\ 1' \otimes X \arrow[ur, "l'_X"'] & \end{tikzcd} \tag{2.12}$$

Stacking the last diagram on top of the second last diagram with $Y = 1'$ and $X = 1$ proves the claim, using that $\eta = l_{1'} \circ (r'_1)^{-1}$.

2.10. **Proof of 2.2.6.** Why is it sufficient to show that “if $b\colon 1 \to 1$ is an isomorphism...”? We want to show that there is a unique isomorphism $\eta\colon 1 \to 1'$ mapping ι to ι' (I explained above what that means). Let $\eta'\colon 1 \to 1'$ be another such isomorphism. Then $b := (\eta')^{-1} \circ \eta\colon 1 \to 1$ is an isomorphism making the diagram (2.15) commutative. And now showing that $b = \mathrm{id}$ solves the problem.

2.11. **Definition 2.2.8.** Very pedantic but l and r are never defined (the l_X and r_X give rise to functors $l\colon \mathcal{C} \to \mathcal{C}$ and $r\colon \mathcal{C} \to \mathcal{C}$).

2.12. **Example 2.3.6.** There was the question why in the definition of the category $\mathcal{C}_G(A)$ we need A to be abelian. The answer is: for $-\otimes-$ to be a bifunctor $\mathcal{C}_G(A) \times \mathcal{C}_G(A) \to \mathcal{C}_G(A)$. The composition in $\mathcal{C}_G(A)$ is the product in A and the tensor product of morphisms is $a \otimes b = ab$. Now, for two morphisms $(a_2, b_1), (a_2, b_1)$ in $\mathcal{C}_G(A) \times \mathcal{C}_G(A)$ we must have

$$
\begin{aligned}
(a_2b_2)(a_1b_1) &= (-\otimes-)(a_2,b_2) \circ (-\otimes-)(a_1,b_1) \\
&= (-\otimes-)((a_2,b_2) \circ (a_1,b_1)) \\
&= (-\otimes-)((a_2a_1, b_2b_1)) \\
&= (a_2a_1)(b_2b_1).
\end{aligned} \tag{2.13}
$$

This is equal if A is abelian.

2.13. **Remark 2.4.2.** I think there's a typo, it should be Section 2.6, not 2.5 (there are no non-trivial monoidal structures discussed in 2.5).

2.14. **After Remark 2.4.2.** If $(F,J)\colon (\mathcal{C}, \otimes, 1) \to (\mathcal{C}^\wr, \otimes^\wr, 1^\wr)$ is a monoidal functor, there is a unique isomorphism $\varphi\colon 1^\wr \to F(1)$ making the diagram

$$
\begin{tikzcd}
1^\wr \otimes^\wr F(1) \arrow[r, "l^\wr_{F(1)}"] \arrow[d, "\varphi\otimes^\wr \mathrm{id}_{F(1)}"'] & F(1) \arrow[d, "F(l_1)^{-1}"] \\
F(1) \otimes^\wr F(1) \arrow[r, "J_{1,1}"'] & F(1 \otimes 1)
\end{tikzcd} \tag{2.14}
$$

commutative. The reason for this is as follows.

(1) If 1 is a unit in a monoidal category $(\mathcal{C}, \otimes)$, then by definition the functor $R_1\colon \mathcal{C} \to \mathcal{C}$ mapping X to $X \otimes 1$ and f to $f \otimes \mathrm{id}_1$ is an equivalence. In particular, the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X\otimes 1, Y \otimes 1)$, $f \mapsto f \otimes \mathrm{id}_1$, is a bijection.
(2) If $\iota\colon 1 \otimes 1 \to 1$ is an isomorphism, so that $(1,\iota)$ is a unit, and if $1' \in \mathcal{C}$ is an object isomorphic to 1 via some isomorphism $\eta\colon 1 \to 1'$, then $(1', \iota')$ with $\iota' = \eta^{-1} \circ \iota \circ (\eta \otimes \eta)$ is also a unit.
(3) Back to the original problem: by assumption $F(1)$ is isomorphic to $1'$, hence it is a unit by 2). By 1) the map $\mathrm{Hom}_{\mathcal{C}^\wr}(1^\wr, F(1)) \to \mathrm{Hom}_{\mathcal{C}^\wr}(1^\wr \otimes F(1), F(1) \otimes^\wr F(1))$, $f \mapsto f \otimes^\wr \mathrm{id}_{F(1)}$, is then a bijection. Now, φ is the preimage of $l^\wr_{F(1)} \circ F(l_1) \circ J_{1,1}$ under this bijection.

2.15. **Exercise 2.4.4.** By definition of φ, the composition

$$
1^\wr \otimes^\wr F(1) \xrightarrow{\varphi \otimes \mathrm{id}_{F(1)}} F(1) \otimes^\wr F(1) \xrightarrow{J_{1,1}} F(1 \otimes 1) \xrightarrow{F(l_1)} F(1) \tag{2.15}
$$

is equal to $l^\wr_{F(1)}$. Apply $-\otimes F(X)$ to this diagram. Using naturality of J and the monoidal structure axiom one obtains the required diagram for $1 \otimes X$. Now, use that $1 \otimes X \simeq X$ to get the diagram for X.

2.16. **Remark 2.4.7.** If $(F,J)\colon \mathcal{C}\to\mathcal{C}'$ and $(F',J')\colon \mathcal{C}'\to\mathcal{C}''$ are monoidal functors, then there is a natural monoidal structure J'' on $F''=F'\circ F$ defined by

$$\begin{tikzcd} F'F(X)\otimes F'F(Y) \arrow[r, "J'_{F(X),F(Y)}"] \arrow[dr, "J''_{X,Y}"'] & F'(F(X)\otimes F(Y)) \arrow[d, "F'(J_{X,Y})"] \\ & F'F(X\otimes Y) \end{tikzcd} \tag{2.16}$$

We consider $F'\circ F$ always with this monoidal structure.

2.17. **Remark 2.4.9.** Applying η to the commutative diagram defining φ yields $\varphi_1^{-1}\otimes^{\wr}\eta_1=(\varphi_2^{-1}\circ\eta_1)\otimes^{\wr}\eta_1$. Composition with $\mathrm{id}_{1^{\wr}}\otimes^{\wr}\eta_1^{-1}$ yields the equality $\varphi_1^{-1}\otimes^{\wr}\mathrm{id}_{F^1(1)}=(\varphi_2^{-1}\circ\eta_1)\otimes^{\wr}\mathrm{id}_{F^1(1)}$. From the above we know that $F^1(1)$ is a unit, hence $R_{F^1(1)}$ is an equivalence, hence this equality implies $\varphi_1^{-1}=\varphi_2^{-1}\circ\eta_1$ as claimed. Note that we use the assumption that η_1 is an isomorphism; we don't get this for free.

I think the reason for asking η_1 to be an isomorphism is the following. It seems natural to require that the diagram

$$\begin{tikzcd} & F^1(1) \arrow[dd, "\eta_1"] \\ 1^{\wr} \arrow[ur, "\varphi^1"] \arrow[dr, "\varphi^2"'] & \\ & F^2(1) \end{tikzcd} \tag{2.17}$$

commutes. As shown on p. 64 in the book by Saavedra, the commutativity of this diagram is in fact equivalent to η_1 being an isomorphism.

2.18. **Remark 2.4.10.** For monoidal equivalences this remark provides way not enough information in my opinion (look at Saavedra's book, Section 4.4). To make things precise, let $(F,J)\colon\mathcal{C}\to\mathcal{C}'$ be a monoidal functor. We call this a monoidal equivalence if there is a monoidal functor $(G,K)\colon\mathcal{C}'\to\mathcal{C}$ together with isomorphisms $\alpha\colon\mathrm{id}_{\mathcal{C}}\to GF$ and $\beta\colon FG\to\mathrm{id}_{\mathcal{C}'}$ of monoidal functors (i.e., morphisms of monoidal functors as defined in the book such that their inverse is again a morphism of monoidal functors; note that we have defined a monoidal structure on the composition and we have a canonical monoidal structure on the identity, so this makes sense). One should call the whole datum $((F,J),(G,K),\alpha,\beta)$ a monoidal equivalence and maybe call (G,K) a monoidal quasi-inverse.

The claim in this remark is now:

Lemma 2.4. *Let $(F,J)\colon\mathcal{C}\to\mathcal{C}'$ be a monoidal functor such that F is an equivalence of ordinary categories. Then any quasi-inverse $G\colon\mathcal{C}'\to\mathcal{C}$ can be "improved" to a monoidal quasi-inverse; in particular, (F,K) is already a monoidal equivalence.*

Proof. Let $X', Y' \in \mathcal{C}'$. Since F is an equivalence, there is a unique morphism $K_{X',Y'} \colon G(X') \otimes' G(Y') \to G(X' \otimes' Y')$ making the diagram

$$\begin{tikzcd} F(G(X') \otimes' G(Y')) \arrow[r, "F(K_{X',Y'})"] \arrow[d, "J_{G(X'),G(Y')}"'] & FG(X' \otimes' Y') \arrow[d, "\beta_{X' \otimes' Y'}"] \\ FG(X') \otimes' FG(Y') \arrow[r, "\beta_{X'} \otimes \beta_{Y'}"'] & X' \otimes' Y' \end{tikzcd} \tag{2.18}$$

commutative. This yields a natural isomorphism $K \colon G(-) \otimes' G(-) \to G(- \otimes' -)$. Moreover, the diagram shows that $\beta \colon FG \to \mathrm{id}_{\mathcal{C}'}$ is a morphism of monoidal functors (for this, recall the definition of the monoidal structure on the composition of monoidal functors from above). So far, so good. But we also need to show that α is monoidal and here I'm not sure. What we can do, while fixing β, is to replace α by another isomorphism, we again denote by α, such that (F, G, α, β) is an adjoint equivalence, meaning that α and β give unit and counit of an adjunction. This is some elementary category theory lemma I think (see here). Now, Lemma 4.4.2.2 in Saavedra's book shows that in this case α is also monoidal. I don't want to type the diagrams used for the proof here... $\square$

I'm unsure whether this works if we do not "improve" the equivalence to an adjoint equivalence. Anyways, the claim in Remark 2.4.10 is true, but this was not "easy to show" for me. Maybe they found a simpler argument?

2.19. **Several problems with Section 2.6.** I don't know what happened here. The title is "Monoidal functors between categories of graded vector spaces", alright. Then it starts with the categories $\mathcal{C}_G^\omega$, why not. But in line 3 it's said that this is the "monoidal category of graded vector spaces introduced in Example 2.3.8". This is not true. Anyways, we can of course consider the same question for the $\mathcal{C}_G^\omega$, so take a monoidal functor $(F, J) \colon \mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$. In the first line of the second paragraph they consider the "restriction to simple objects". Again, it looks like they actually wanted to talk about Vec_G^ω instead. Anyways, it's still true that F is a group morphism $G_1 \to G_2$. But now the equation after (2.30) giving the monoidal structure axiom is confusing. On the very left there is $\omega_1(g, h, l)$. I think this should actually be $F(\omega_1(g, h, l))$. It seems to me they implicitly assume that F is the identity on morphisms and I don't know why this should be the case. The statement further down below that monoidal functors correspond to pairs (f, μ) thus seems to be incorrect; there should be an additional datum of an endomorphism on A in my opinion, not?

Let's instead turn to the title of this section: k-linear monoidal functors $(F, J) \colon k\text{-}\mathrm{Vec}_G^\alpha \to k\text{-}\mathrm{Vec}_H^\beta$, where G and H are groups. I claim that F defines a (necessarily unique) group morphism $f \colon G \to H$ such that $F(k_g) \simeq k_{f(g)}$ for all $g \in G$. Since F is monoidal, we have $F(k_1) \simeq k_1$. We have $k_g \otimes k_{g^{-1}} = k_1$. Since F is monoidal, we thus have $F(k_g) \otimes F(k_{g^{-1}}) \simeq F(k_g \otimes k_{g^{-1}}) \simeq F(k_1) = k_1$. It follows that $\dim F(k_g) = 1$, so $F(k_g) \simeq k_{f(g)}$ for some $f(g) \in H$. It's clear that $g \mapsto f(g)$ defines a group morphism $f \colon G \to H$.

Now comes the fun fact. For any $g \in G$ our functor F defines a map $F_g \colon k = \mathrm{End}(k_g) \to \mathrm{End}(k_{f(g)}) = k$. The endomorphism sets are k-algebras by k-linearity of the category. Since we assume F to be k-linear, F_g is a k-algebra morphism. Hence, F_g must be the identity! The monoidal structure axiom in this case looks indeed like the one written down in Section 2.6, since $F(\omega_1(g,h,l)) = \omega_1(g,h,l)$, and k-linear monoidal functors correspond to pairs (f,μ) as claimed.

2.20. **Proof of Theorem 2.8.5.** It is shown that $L\colon \mathcal{C} \to \mathcal{C}^l$ is a monoidal functor and that it is an equivalence of ordinary categories. Hence, by the remark above, it is already a monoidal equivalence.

2.21. **Remark 2.8.6.** A monoidal category which is monoidal isomorphic to a strict one is also strict. This is why $\mathcal{C}_G^\omega(A)$ is not isomorphic to a strict monoidal category.

2.22. **Typo in Exercise 2.9.1.** It's the $(n-1)$-st Catalan number C_{n-1}.

2.23. **Remark 2.10.3.** I think the equation ${}^*(X^*) \simeq X \simeq ({}^*X)^*$ implicitly assumes uniqueness of left and right dual, which is proven only later in Proposition 2.10.5.

2.24. **Typo in Remark 2.10.9.** in the last parenthesis it should be $-\otimes {}^*V$, not $-\otimes V^*$.

2.25. **Example 2.10.2.** The "contraction" $\mathrm{ev}_V : V^* \otimes V \to \Bbbk$ is the map $f \otimes v \mapsto f(v)$; the "usual embedding" $\Bbbk \to V \otimes V^*$ is the map $1 \mapsto \sum_i v_i \otimes v_i^*$, where (v_i) is a basis of V with dual basis (v_i^*), this being independent of the choice of basis.

2.26. **Typos in paragraph after Example 2.10.14.** It is by Exercise 2.10.7(b), not 2.10.7(ii). The same paragraph: we have $\mathrm{Hom}(W^*, V^*) \simeq \mathrm{Hom}(W^* \otimes 1, V^*) \simeq \mathrm{Hom}(1, W \otimes V^*) \simeq \mathrm{Hom}(1 \otimes V, W) \simeq \mathrm{Hom}(V, W)$ by (2.50) and (2.49). This shows that the functor $(-)^*$ is fully faithful.

2.27. **Example 2.12.6.** "A monoidal category is the same thing as a 2-category with one object." I think here one needs to add "strict" or "up to monoidal equivalence".

3. Chapter 3 : $\mathbb{Z}_+$-rings

3.1. **First definitions in Section 3.1.** Several new terms are introduced; here's a summary:

- A $\mathbb{Z}_+$-ring is a ring A with a fixed $\mathbb{Z}$-basis which has non-negative structure constants and such that 1 is a non-negative linear combination of the basis elements.
- A $\mathbb{Z}_+$-ring is unital if 1 itself is a basis element.
- A $\mathbb{Z}_+$-ring A is based if there's an involution $*$ on the basis which extends to a ring anti-involution of A and $\tau(b_i b_j) = \delta_{i,i^*}$, where $\tau(b_i) = \delta_{i \in I_0}$. The latter condition means that in any product $b_i b_j$ there is no basis element from I_0 if $j \neq i^*$ and if $j = i^*$ there is a unique basis element from I_0, which then occurs with multiplicity 1.
- A multifusion ring is a based ring with a finite basis.
- A fusion ring is a unital multifusion ring.

3.2. **Exercise 3.1.5(i).** Let $1 = \sum_j a_j b_j$. Then

$$b_i = b_i \cdot 1 = \sum_j a_j b_i b_j = \sum_j a_j \left(\sum_k c_{ij}^k b_k \right) = \sum_k \left(\sum_j a_j c_{ij}^k \right) . \tag{3.1}$$

Hence,

$$\sum_j a_j c_{ij}^k = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

For $j \in I_0$ we have $a_j > 0$ and so we must have $c_{ij}^k = 0$ if $j \in I_o$ and $k \neq i$. Hence, $b_i b_j = c_{ij}^i b_i$ for $j \in I_0$. In particular, $b_i^2 = c_{ii}^i b_i$ and since $a_i > 0$, it follows from the equation above that $a_i = 1 = c_{ii}^i$. In particular, $b_i^2 = b_i$. Moreover, it then follows that $c_{ij}^i = 0$ for $i \neq j \in I_0$, so $b_i b_j = 0$.

Note that we have shown that $a_i = 0$ for all $i \in I_0$, i.e., $1 = \sum_{i \in I_0} b_i$. Hence, Proposition 3.1.4 holds actually for any $\mathbb{Z}_+$-ring already.

3.3. **Exercise 3.1.5(ii).** Let A be a $\mathbb{Z}_+$-ring. To make this into a based ring we need an involution $*$. But there is at most one choice for this. We need to look at the multiplication table of the $b_i b_j$. For each i there must be a unique index $i^* \in I$ such that $b_i b_i^*$ contains a single basis element from I_0, and this with multiplicity 1. Then only $i \mapsto i^*$ can be the involution of a based algebra (and what remains to check is that this is an anti-involution). If the multiplication table does not have this property, the ring cannot be based. Hence, the involution is a property, not an additional structure.

3.4. **Example 3.1.9(iv).** The reason that the ring R_G of complex representations of a finite group G is a fusion ring is the following. We have $(V \otimes W)^* \simeq W^* \otimes V^*$ by Exercise 2.10.7(b). Hence, $(-)^*$ is an anti-involution on R_G. If V and W are two irreducible representations, then the multiplicity of $\mathbb{C}$ in $V \otimes W$ is

$$[V \otimes W : \mathbb{C}] = \dim \mathrm{Hom}_G(\mathbb{C}, V \otimes W) = \dim \mathrm{Hom}_G(V^*, W) . \tag{3.3}$$

Since V^* and W are simple, this is zero whenever V^* is not isomorphic to W. If V^* is isomorphic to W, then this multiplicity is equal to 1 since $\mathbb{C}$ is algebraically closed. Hence, the property for fusion rings is satisfied.

Note that for the formula relating the multiplicity with the dimension of the Hom-space we need the trivial representation $\mathbb{C}$ to be projective. The same is true over an arbitrary field $\Bbbk$ if and only if $\mathrm{Rep}_{\Bbbk}(G)$ is semisimple, so, if and only if the characteristic of $\Bbbk$ does not divide the order of G. For example, as mentioned in Example 3.1.9(v), for S_3 over any field of characteristic 2 (algebraically closed or not) the product $V \otimes V^*$ with the 2-dimensional irreducible representation V contains two copies of the trivial representation.

3.5. **Example 3.1.9(v).** Where is the reference for this?

3.6. **Definition 3.3.1.** Think of "transitive" in the way that the "orbit" of any $X \in I$ under the action of I by left/right multiplication is all of I, where "orbit" means we collect all the basis elements in the product.

3.7. **Exercise 3.3.2.** We begin with:

Lemma 3.1. *Let A be a unital based algebra with basis I. Then $XZ \neq 0$ for any $X, Z \in I$.*

Proof. Suppose that $XZ = 0$. Then also $X^*XZ = 0$. Since A is unital and based, we have

$$
\begin{aligned}
(X^*X)Z &= \left(1 + \sum_{Y \neq 1} c^Y_{X^*,X} Y\right) Z \\
&= Z + \sum_{Y \neq 1} c^Y_{X^*,X} \sum_U c^U_{Y,Z} U \\
&= (1 + \sum_{Y \neq 1} c^Y_{X^*,X} c^Z_{Y,Z}) Z + \sum_{U \neq Z} \left(\sum_{Y \neq 1} c^Y_{X^*,X} c^U_{Y,Z} \right) U.
\end{aligned} \tag{3.4}
$$

But $1 + \sum_{Y \neq 1} c^Y_{X^*,X} c^Z_{Y,Z} \neq 0$ since all the coefficients are non-negative. Hence, $X^*XZ \neq 0$, a contradiction. □

Now, to the exercise. From the lemma we know that $X^*Z \neq 0$. Hence, there is $Y_1 \in I$ with $c^{Y_1}_{X^*,Z} \neq 0$. We have $c^{Y_1}_{X^*,Z} = c^{Z^*}_{Y_1^*,X^*} = c^Z_{X,Y_1}$, hence, Z occurs in XY_1. This shows that A is transitive.

In generalization of this we have:

Lemma 3.2. *If A is unital and transitive, then $XZ \neq 0$ for any $X, Z \in I$.*

Proof. Since A is unital and transitive, we have $XZ \neq 0$ for any $X, Z \in I$. Namely, suppose that $XZ = 0$. We can find $Y \in I$ with $ZY = a1 + \sum_{U \neq 1} c^U_{Z,Y} U$ with $a > 0$ since A is unital and transitive. We then get

$$
\begin{aligned}
0 = XZY &= aX + \sum_{U \neq 1} c^U_{Z,Y} XU \\
&= aX + \sum_{U \neq 1} \sum_V c^U_{Z,Y} c^V_{X,U} V \\
&= \left(a + \sum_{U \neq 1} c^U_{Z,Y} c^X_{X,U} \right) X + \sum_{V \neq X} \left(\sum_{U \neq 1} c^U_{Z,Y} c^V_{X,U} \right) V.
\end{aligned} \tag{3.5}
$$

The coefficient of X is non-zero, so $XYZ \neq 0$, a contradiction. □

3.8. **Frobenius-Perron dimension.** For the definition of the Frobenius–Perron dimension we just need a $\mathbb{Z}_+$-ring A. For the properties in Proposition 3.3.6 we then need unital, transitive, and of finite rank.

4. Chapter 4: Tensor categories

4.1. **Definitions of Section 4.1 and 4.2.** Again a lot of definitions. Here's a summary:

- A multiring category is a $\Bbbk$-linear, abelian, locally finite (locally finite means finite-dimensional Hom-spaces and all objects have finite length), monoidal category with bilinear and biexact tensor product. If in addition $\mathrm{End}_{\mathcal{C}}(1) = \Bbbk$, it's called a ring category.

- A (multi)tensor category is a (multi)ring category which is also rigid (using Proposition 4.2.1; I hope that's correct).
- A (multi)fusion category is a (multi)tensor category which is also semisimple and finite (finite means that it is locally finite, has enough projectives, and finitely many simples up to isomorphism).
- A quasi-tensor functor $F\colon \mathcal{C} \to \mathcal{D}$ between multiring categories is an exact and faithful linear functor equipped with a functorial isomorphism $J\colon F(-)\otimes F(-) \to F(- \otimes -)$ and $F(1) \simeq 1$. It is a tensor functor if it is monoidal, i.e., J satisfies the coherence diagram. The assumption that the functor is exact and faithful seem to be imposed only in this book, not in the general literature.

Moreover: the Grothendieck ring of a multiring category $\mathcal{C}$ is a $\mathbb{Z}_+$-ring. It is unital if and only if $1 \in \mathcal{C}$ is simple, e.g., if $\mathcal{C}$ has left duals. It is based if $\mathcal{C}$ is a semisimple multitensor category. It is a (multi)fusion ring if $\mathcal{C}$ is a (multi)fusion category.

4.2. **Proof of Proposition 4.2.1.** Note that exactness implies additivity already.

4.3. **Proof of Proposition 4.2.8.** Note that $0 \otimes I_2 = 0$ by exactness of the tensor product.

4.4. **Proof of Proposition 4.2.10.** Frobenius–Perron dimension for monoidal categories is at this stage not yet defined. I think this proposition needs to come after Section 4.5.

4.5. **Proof of Corollary 4.2.13.** This uses the elementary fact that a locally finite abelian category is semisimple if and only if all objects are projective, see maybe here.

4.6. **Proof of Theorem 4.3.1.** Two elementary comments. Since $\mathrm{End}_{\mathcal{C}}(1)$ is finite-dimensional by locally finiteness, it's an artinian ring, so it's semisimple if and only if its Jacobson radical is zero. The Jacobson radical is equal to the nilradical since the ring is artinian. Hence, it is semisimple if and only if it has no non-zero nilpotent elements. Suppose that $a^2 = 0$ implies $a = 0$ for any a. By induction this shows that the ring has no nilpotent elements. Namely, let $a^n = 0$. If n is even, we can write $0 = a^n = (a^{\frac{n}{2}})^2$, so $a^{\frac{n}{2}} = 0$ and then $a = 0$ by induction. If n is odd, then $0 = a^{n+1} = (a^{\frac{n+1}{2}})^2$, so $a^{\frac{n+1}{2}} = 0$ and then $a = 0$ by induction.

The other comment: that $K \otimes 1$ is a subobject of $1 \otimes 1$ follows from the exactness of the tensor product.

4.7. **Remark 4.3.4.** It's stated in at the end of Section 4.5 but it may be helpful to note here already that decomposition can be regarded as an analogue of the Pierce decomposition of a ring.

4.8. **Theorem 4.4.1.** The formulation "with simple object 1" is a bit strange; it should more precisely be "such that the unit object 1 is simple".

4.9. **Proof of Theorem 4.4.1.** Let V be a self-extension of 1, so there's an exact sequence $0 \to 1 \to V \to 1 \to 0$. In particular, the only constituent of V is 1, and this occurs with multiplicity 2. Hence,

$$2 = [V : 1] = \dim \mathrm{Hom}(P(1), V) \ , \tag{4.1}$$

this formula being standard for finite-dimensional algebras.

At the moment I don't see why the proof doesn't work with an arbitrary simple object instead of the unit object (I'm sure it doesn't).

4.10. **Remark 4.5.6.** "Then F defines a homomorphism of unital $\mathbb{Z}_+$-rings...". I think the unital is wrong here, the categories are just multiring categories, so 1 may not be simple. I think it should be "...defines a unital homomorphism of $\mathbb{Z}_+$-rings..." if there's a need to include unital at all.

4.11. **Definition 4.7.11.** I don't know why there's a Tr^L in the sentence.

5. Chapter 5: Representation categories of Hopf algebras

5.1. **Before Definition 5.2.3.** "Moreover, the forget functor $\mathrm{Rep}(H) \to \mathrm{Vec}$ is a fiber functor." A fiber functor is only defined on a ring category. Hence, one first needs to show that $\mathrm{Rep}(H)$ is a ring category.

5.2. **Reminder Hopf algebra.** Here's a summary (or reminder) of some Hopf algebra constructions. Let $H = (H, \mu, i, \Delta, \epsilon, S)$ be a Hopf algebra over a field k. The following are again Hopf algebras (Exercises 5.2.5, 5.3.17 and 5.3.19):

$$\begin{aligned} H_{op} &= (H, \mu^{op}, i, \Delta, \epsilon, S^{-1}), \\ H^{cop} &= (H, \mu, i, \Delta^{op}, \epsilon, S^{-1}), \\ H^{cop}_{op} &= (H, \mu^{op}, i, \Delta^{op}, \epsilon, S), \\ H^* &= (H^*, \Delta^*, \epsilon^*, \mu^*, i^*, S^*) \end{aligned} \tag{5.1}$$

if H is finite-dimensional Moreover, if H and G are Hopf algebras, then so is $H \otimes_k G$ with the following data:

$$\begin{aligned} &\mu_{H\otimes G} = (\mu_H \otimes \mu_G) \circ (Id \otimes \tau \otimes Id) : H \otimes G \otimes H \otimes G \to H \otimes G \\ &\eta_{H\otimes G} = (\eta_H \otimes \eta_G) \circ \phi^{-1} : k \to H \otimes G \\ &\Delta_{H\otimes G} = (Id \otimes \tau \otimes Id) \circ (\Delta_H \otimes \Delta_G) : H \otimes G \to H \otimes G \otimes H \otimes G \\ &\varepsilon_{H\otimes G} = \phi \circ (\varepsilon_H \otimes \varepsilon_G) : H \otimes G \to k \\ &S_{H\otimes G} = S_H \otimes S_G : H \otimes G \to H \otimes G \end{aligned} \tag{5.2}$$

6. Chapter 7: Module categories

6.1. **Exercise 7.3.2.** A multitensor category is rigid, so has duals. Proposition 7.1.6 now implies that for fixed $X \in \mathcal{C}$ the functor $X \otimes -$ is right adjoint to $X^* \otimes -$ and left adjoint to ${}^*X \otimes -$. Hence, $X \otimes -$ is exact.

6.2. **Proposition 7.3.3.** Before the proposition it is argued that the category $\mathrm{End}_l(\mathcal{M})$ of left exact endofunctors on $\mathcal{M}$ is abelian. Actually, an argument is only given if $\mathcal{M}$ is finite-dimensional comodules over a coalgebra. I don't understand this argument yet (especially the ind-completion part, I don't even know what this is...). Anyways, believing this, what about arbitrary $\mathcal{M}$? I think here we want to use this result by Takeuchi from 1.9/1.10 which says that any essentially small locally finite abelian category over a field is equivalent to the category of finite-dimensional comodules over a coalgebra. So, for essentially small categories this seems to be fine. But without this assumption I don't see how it works. I guess, it's assumed without mentioning.

Also, I'm sure that one needs to consider additive endofunctors on $\mathcal{M}$ because the action $\otimes\colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear.

6.3. **Example 7.4.6.** The "Obviously, $\mathcal{M} = \mathrm{End}(\mathcal{M})$" is too quick for me. Let $\mathcal{M} = \mathrm{Vec}$. By the enriched/linear Yoneda lemma (reference?) the map $V \mapsto \mathrm{Hom}_{\mathcal{M}}(V, -)$ is an embedding $\mathcal{M} \to \mathrm{End}(\mathcal{M})$, the latter category being linear endofunctors on $\mathcal{M}$. By elementary linear algebra we have an isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(V \otimes W, Z) \simeq \mathrm{Hom}_{\mathcal{M}}(V, \mathrm{Hom}_{\mathcal{M}}(W, Z)) \ . \tag{6.1}$$

Hence, the embedding maps $V \otimes W$ to $\mathrm{Hom}_{\mathcal{M}}(V, -) \circ \mathrm{Hom}_{\mathcal{M}}(W, -)$. Recall that $\mathrm{End}(\mathcal{M})$ is monoidal with tensor product being the composition. The canonical isomorphisms above now give a monoidal structure on the embedding $\mathcal{M} \to \mathrm{End}(\mathcal{M})$. The essential image of this embedding is the category $\mathrm{End}_l(\mathcal{M})$ of left exact linear functors. Hence, given a tensor category $\mathcal{C}$, the $\mathcal{C}$-module structures on Vec correspond to tensor functors $\mathcal{C} \to \mathrm{Vec}$.

6.4. **Definition 7.5.1.** By definition a module category $\mathcal{M}$ is already locally finite, so maybe this addition in the definition is not necessary.

6.5. **Paragraph after Definition 7.14.1.** Too quick for me! Okay, we can define R_{XZ} via $\gamma_X = \sigma \circ R_{XZ}$ where σ is the permutation on the factors. But the next two statements were not clear to me.

The second statement is that "R_{HZ} commutes with the right multiplication by elements in the first component." Let's see why. For $h \in H$ let $\rho_h\colon H \to H$ be right multiplication by h. Then

$$\sigma \circ (\rho_h \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes \rho_h) \circ \sigma \ . \tag{6.2}$$

Moreover, by functoriality of γ_H applied to ρ_h the diagram

$$\begin{tikzcd} H \otimes Z \arrow[r, "\gamma_H"] \arrow[d, "\rho_h \otimes \mathrm{id}_Z"'] & Z \otimes H \arrow[d, "\mathrm{id}_Z \otimes \rho_h"] \\ H \otimes Z \arrow[r, "\gamma_H"] & Z \otimes H \end{tikzcd} \tag{6.3}$$

commutes, i.e.,

$$\gamma_H \circ (\rho_h \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes \rho_h) \circ \gamma_H \ . \tag{6.4}$$

Applying σ^{-1} to this yields

$$\sigma^{-1} \circ \ \gamma_H \circ (\rho_h \otimes \mathrm{id}_Z) = \sigma^{-1} \circ (\mathrm{id}_Z \otimes \rho_h) \circ \gamma_H \ , \tag{6.5}$$

hence

$$R_{HZ} \circ (\rho_h \otimes \mathrm{id}_Z) = (\rho_h \otimes \mathrm{id}_Z) \circ \sigma^{-1} \circ \gamma_H = (\rho_h \otimes \mathrm{id}_Z) \circ R_{HZ} \ . \tag{6.6}$$

This is precisely the claim.

6.6. **Exercise 7.14.2.** The symbols R^{12}, R^{13}, and R^{23} are actually never defined in the book. I believe they come from the following. We fix two k-algebras X, Y and an element $R \in X \otimes Y$. We fix another k-algebra Z. We then have the following three maps:

$$\phi_{12}\colon X \otimes Y \to X \otimes Y \otimes Z\ , \quad x \otimes y \mapsto x \otimes y \otimes 1 \tag{6.7}$$

$$\phi_{13}\colon X \otimes Y \to X \otimes Z \otimes Y\ , \quad x \otimes y \mapsto x \otimes 1 \otimes y \tag{6.8}$$

$$\phi_{23}\colon X \otimes Y \to Z \otimes X \otimes Y\ , \quad x \otimes y \mapsto 1 \otimes x \otimes y\ . \tag{6.9}$$

Now, define

$$R^{12} := \phi_{12}(R)\ , \quad R^{13} := \phi_{13}(R)\ , \quad R^{23} := \phi_{23}(R)\ . \tag{6.10}$$

7. Chapter 8: Braided categories

7.1. **After Definition 8.1.7.** I think —enhancing my comment for Section 2 above— one can show that if a $(F, J)\colon \mathcal{C} \to \mathcal{C}'$ is a braided monoidal functor such that F is an equivalence of categories, then any quasi-inverse of F is also braided monoidal.

7.2. **Enhancing Proposition 8.1.10.** A braiding on a strict monoidal category is precisely a c satisfying the Yang–Baxter equation (8.6).

7.3. **Section 8.3.** The correspondence between R-matrices and braidings is unfortunately not so nicely carried out in my opinion (I was actually a bit lost). There are really wonderful lecture notes by C. Schweigert [8] where all this is explained completely and nicely. Here's a summary. First of all, recall from my comments to Section 5 that if A is a bialgebra, then so is $A \otimes A$. If $R = R^{(1)} \otimes R^{(2)}$ is an element of $A \otimes A$ (in Sweedler notation), there are three ways to view it as an element of $A \otimes A \otimes A$:

$$R^{12} := R^{(1)} \otimes R^{(2)} \otimes 1\ , \quad R^{13} := R^{(1)} \otimes 1 \otimes R^{(2)}\ , \quad R^{23} := 1 \otimes R^{(1)} \otimes R^{(2)}\ . \tag{7.1}$$

Theorem 7.1. *Let A be a bialgebra. Then the tensor category* $\mathrm{Rep}(A)$ *is braided if and only if A is quasi-triangular, i.e. there is an invertible element $R \in A \otimes A$ such that*

$$(\Delta \otimes \mathrm{id}_A)(R) = R^{13}R^{23}\ , \quad (\mathrm{id}_A \otimes \Delta)(R) = R^{13}R^{12}\ , \quad \Delta^{op}(a) = R\Delta(a)R^{-1}\ \forall a \in A\ . \tag{7.2}$$

Both structures (braidings and such elements R) are in one-to-one correspondence.

Remark 7.2. The element R is called universal R-matrix. The last condition is called quasi-cocommutativity. Schweigert [8] says: "There is no universally accepted definition for the term quantum group. I would prefer to use the term for quasi-triangular Hopf algebras. Some authors use it as a synonym for Hopf algebras, some for certain subclasses of quasi-triangular Hopf algebras."

Proof. Let (A, R) be quasi-triangular. We need to construct a braiding on $\mathrm{Rep}(A)$ from R. For $U, V \in \mathrm{Rep}(A)$ define

$$\begin{aligned} c_{U,V}\colon U \otimes V &\to V \otimes U \\ u \otimes v &\mapsto \sigma(R(u \otimes v)) = R^{(2)}v \otimes R^{(1)}u\ . \end{aligned} \tag{7.3}$$

Here, σ is the flip in the components. One can check that this is indeed a morphism of A-modules. Since R is invertible, this map is invertible. One can now prove the hexagon axioms by direct computation (see [8]). Hence, c is a braiding on $\mathrm{Rep}(A)$. Conversely, assume that we are given a braiding c on $\mathrm{Rep}(A)$. We define the element R by

$$R := \sigma \circ c_{A,A}(1 \otimes 1) \in A \otimes A \; . \tag{7.4}$$

One can now show by direct computation (see [8]) that c is defined by R exactly as above. Moreover, from the hexagon axioms one obtains the first two relations on R, the last one (quasi-cocommutativity) follows essentially from A-linearity of c. $\square$

7.4. **Section 8.9.** The assumption for the whole section is that monoidal categories are strict. (I don't agree with the second sentence "Equivalently, we suppress all associativity and unit constraints.": suppressing notation doesn't make things equivalent). However, already the first example (representation categories) is not strict. I think it's fine but still... For example, what happens to the relation (8.31)

$$u_X \otimes u_Y = u_{X \otimes Y} \circ c_{Y,X} \circ c_{X,Y} \tag{7.5}$$

in the non-strict case? Is it the same? Moreover, one needs to assume throughout the whole section (and also later) that $\mathcal{C}$ has duals.

7.5. **Proposition 8.10.12.** I think this needs to be $\psi = u \circ \theta$, not $\psi = \theta \circ u$.

7.6. **After Proposition 8.10.12.** Maybe the trace should be recalled before proving the proposition as it's used in the proof.

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