1) Suppose that $P$ is non-split and unramified (then are only finitely many ramified primes by lecture, so if there are only finitely many split + unramified, then are only finitely many non-split).

Let $Q$ be the prime above $P$.

Since $P$ non-split, $G_Q = G$

Fundamental equation is

$|G| = [C : k] = e f r = f$ (using non-split + unramified)

$\Rightarrow |G_Q| = f$

We also have

$f = [G_Q : \mathbb{Q}] \Rightarrow |G_Q| = 1 \Rightarrow G_Q \cong \text{Gal}_{k(Q)}(k(Q))$

This is a contradiction since $\text{Gal}_{k(\mathbb{Q})}(k(Q))$ is cyclic but $G_Q \cong G$ is not.

2) a) Since $Q$ unramified, $|G_Q| = 1$ \Rightarrow $G_Q \cong \text{Gal}_{k(\mathbb{Q})}(k(Q))$

$k(\mathbb{Q}) \subset k(Q)$ is an extension of finite fields

$\Rightarrow \text{Gal}_{k(\mathbb{Q})}(k(Q))$ is cyclic and generated by the automorphism

$x \mapsto x^q$, $q = \#k(Q)$, $x \in G_Q$

Thus define $\sigma_Q$ as the corresponding element in $G_Q$.

Suppose $\sigma \in G$ is any other automorphism with this property, i.e.,

$\sigma(x) \equiv x^q \text{ mod } \mathbb{Q}$

Claim: $\sigma \in G_Q$.

Proof: Suppose $\sigma \not\in G_Q$ \Rightarrow $\sigma Q + Q \Rightarrow \exists x \in Q$ with $\sigma(x) \not\in Q$

$\Rightarrow \sigma(x) \equiv 0 \text{ mod } \mathbb{Q}$ \Rightarrow $x^q \equiv 0 \text{ mod } \mathbb{Q}$ \Rightarrow $x \equiv 0$ \Rightarrow $x \in \mathbb{Q}$. 

Uniqueness: \(-G_\mathfrak{p}\) clear now.

b) \(P\) totally split means \(r = n\), \(t_i = 1 = e_i\)

\(-|\mathbb{I}_{\mathbb{Q}}| = e = 1, [\mathbb{G}_{\mathbb{Q}}, \mathbb{I}_{\mathbb{Q}}] = t = 1\)

\(-\mathbb{G}_{\mathbb{Q}}\) is trivial \(-\left(\frac{\mathbb{U}_{\mathbb{Q}}}{\mathbb{Q}}\right)\) is trivial.

Conversely, suppose \(-\left(\frac{\mathbb{U}_{\mathbb{Q}}}{\mathbb{Q}}\right)\) is trivial.

Since \(P\) unramified \(-e = 1 = \mathbb{I}_{\mathbb{Q}}\) is trivial.

\(-\mathbb{G}_{\mathbb{Q}} = \mathbb{G}_{\mathbb{Q}, \mathfrak{p}}(\mathfrak{p}(\mathbb{Q}))\) is trivial.

\(-t = 1\)

\(-P\) splits completely.

c) \(\text{Have } \mathbb{G}_{\mathbb{Q}} = \mathbb{Q}\) for \(\text{all } \mathfrak{a} \in \mathbb{G}\).

\(\mathbb{G}_{\mathbb{Q}}(x) = x^{\mathfrak{a}(\mathbb{Q})} \mod \mathbb{Q} \quad \forall x \in \mathbb{Q}

\(-\mathbb{I} \mathbb{G}_{\mathbb{Q}}(x) = \mathbb{I}(x^{\mathfrak{a}(\mathbb{Q})}) \mod \mathbb{Q}\)

\(-\mathbb{I} \mathbb{G}_{\mathbb{Q}}(x) = \mathbb{I}(x)^{\mathfrak{a}(\mathbb{Q})} \mod \mathbb{Q}\)

This holds for all \(x\), so also for \(\mathbb{I}^{-1}(x)\).

\(-\mathbb{I} \mathbb{G}_{\mathbb{Q}}(\mathbb{I}^{-1}(x)) = x^{\mathfrak{a}(\mathbb{Q})} \mod \mathbb{Q}\)

\(-\mathbb{I} \mathbb{G}_{\mathbb{Q}} \mathbb{I}^{-1} = \mathbb{G}_{\mathbb{Q}}\)