6.5 Discreteness of Lattices

Back to a general lattice $\Lambda$ in $\mathbb{R}^n$, considered with the standard basis $e_1, \ldots, e_n$ and standard scalar product $\langle \cdot, \cdot \rangle$.

Let $v_1, \ldots, v_n$ be a basis of $\Lambda$, $Q = \langle v_1, v_j \rangle = (v_i, v_j)$, and $v_i = \sum a_{ij} e_i$, so $Q = AA^T$.

Let $x \in \Lambda$, so $x = \sum c_i v_i$. Then

$$\|x\|^2 = \langle x, x \rangle = \sum c_i^2 <v_i, v_j> = \sum c_i^2 Q_{ij} = x^T Q x = : Q(x)$$

For a constant $C > 0$ we are interested in

$$\{ x \in \mathbb{Z}^n | \|x\|^2 \leq C \}.$$

So, we need to find lattice point inside the ellipsoid

$$\{ x \in \mathbb{R}^n | Q(x) \leq C \}.$$

Let $\tilde{Q}$ be the quadratic supplement of $Q$, so

$$Q(x) = \sum_{i=1}^n \tilde{Q}_{ii} \left( x_i + \sum_{j=i+1}^n \tilde{Q}_{ij} x_j \right)^2.$$

Then

$$Q(x) \leq C \iff \tilde{Q}_{ii} \left( x_i + \sum_{j=i+1}^n \tilde{Q}_{ij} x_j \right)^2 \leq C - \sum_{j=1}^{n-1} \tilde{Q}_{jj} x_j^2 = : \tilde{C} \in \mathbb{R}^n$$

for $i = n, n-1, \ldots, 1$.

Now, do a backtrack search:

1. Find the $x_n \in \mathbb{Z}$ with $|x_n| \leq \sqrt{\tilde{C}/\tilde{Q}_{nn}} = \sqrt{C/\tilde{Q}_{nn}}$

2. For fixed $x_{i+1}, \ldots, x_n \in \mathbb{Z}$ satisfying

$$\sum_{j=i+1}^n \tilde{Q}_{jj} x_j^2 \leq \tilde{C}_{i+1}$$
determine all possibilities for $x_i$ as follows

$$U_i := \sum_{j=i+1}^{\infty} \hat{a}_j x_j \text{ for } n \geq i \geq 1$$

and then find the $x_i$ with

$$-\sqrt{\frac{T_i}{\hat{a}_i^2}} - U_i \leq x_i \leq \sqrt{\frac{T_i}{\hat{a}_i^2}} - U_i$$

This is a constructive algorithm and it is clear that:

**Corollary 6.17**

1. For each $C > 0$ there are only finitely many $x \in \Lambda$ with $\|x\| \leq C$.
2. $\Lambda$ is a discrete subset of $\mathbb{R}^n$.
3. If $(x_m)_{m \in \mathbb{N}}$ is a sequence in $\Lambda$ which converges to $x \in \mathbb{R}^n$, then $x \in \Lambda$.  \(\Box\)

### 6.6 Shortest vectors and lattice density

(6.17) implies that $\Lambda$ contains a **shortest non-zero vector**. By $\S 6.4$ we have an algorithm to find one. Let $\lambda_{\times} (\Lambda)$ be the length of the shortest vectors.

This quantity is related to the density of $\Lambda$.

For $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$, let $B^n(x, r) = \{ y \in \mathbb{R}^n \mid \|x-y\| \leq r \}$ be the ball of radius $r$ centered at $x$. A **sphere packing** is a collection

$$P = \bigcup_{x \in X} B^n(x, r)$$

for some set $X \subset \mathbb{R}^n$ such that the balls have pairwise disjoint interior.

The **density** $\rho(P)$ of $P$ quantifies how much of the volume of $\mathbb{R}^n$ is made up of $P$; precisely:

$$\rho(P) = \lim_{t \to \infty} \frac{\text{Vol}(P \cap B^n(t))}{\text{Vol}(B^n(t))} \quad \text{for all } t \in \mathbb{R}$$

If $x = \Lambda$ is a lattice, then $P$ is called a **lattice sphere packing**, e.g.
Since $\Lambda$ is additive, we have

$$\chi_\Lambda(\Lambda) = \min_{x \in \Lambda} \|x\| = \min_{x,y \in \Lambda} \|x-y\|$$

Hence, the maximal radius for the balls of a lattice sphere packing with $X=\Lambda$ is $\frac{1}{2} \chi_\Lambda(\Lambda)$. The corresponding density $g(\Lambda)$ is the density of $\Lambda$.

This can be computed relative to the volume of the fundamental region.

Recall from §6.2 that

$$d(\Lambda) = \sqrt{\sum_{i,j} b_{ij}} = \text{vol}(\phi)$$

is independent of the chosen basis.

By symmetry you can see that

$$g(\Lambda) = \frac{\text{vol}(B(\frac{1}{2} \chi_\Lambda(\Lambda)))}{\text{vol}(\phi)}$$

We have

$$\text{vol}(B^n(r)) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n = \text{vol}(B^n(1)) \cdot r^n$$

Euler Gamma

$$\Rightarrow g(\Lambda) = \frac{\chi_\Lambda(\Lambda)^n}{d(\Lambda)} \cdot 2^{-n} \nu_n$$

is the density of $\Lambda$.

For fixed $n$, what is the maximal density one can achieve with a lattice sphere packing? This amounts to knowing
\[ \varrho_n := \text{Sup} \frac{\varrho(\Lambda)}{\text{lattice}} \iff \sqrt[\varrho_n] := \text{Sup} \frac{\chi_\Lambda(\Lambda)}{d(\Lambda)^{1/n}} \]

*\( \delta_n \) is called Hermite constant*

\[ \Rightarrow \varrho_n = \sqrt[\varrho_n] \cdot 2^{-n} \chi_n \]

This is only known in a few cases:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9 ≤ ( n ) ≤ 23</th>
<th>24</th>
<th>( n \geq 25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_n )</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>64/3</td>
<td>64</td>
<td>256</td>
<td>?</td>
<td>4.24</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( \approx \varrho_n )</td>
<td>1</td>
<td>0.307</td>
<td>0.74</td>
<td>0.617</td>
<td>0.465</td>
<td>0.373</td>
<td>0.295</td>
<td>0.254</td>
<td>?</td>
<td>0.002</td>
<td>?</td>
</tr>
</tbody>
</table>

**FACT:** \( \gamma_n \) is a rational number,

Keller's conjecture (general packings)

The sphere packing interpretation immediately gives us an upper bound for \( \chi_\Lambda(\Lambda) \):

\[ \varrho_n \leq \frac{1}{\sqrt{3}} \Rightarrow \frac{\chi_\Lambda(\Lambda)^n}{d(\Lambda)} \cdot 2^{-n} \chi_n \leq 1 \]

\[ \Rightarrow \frac{\chi_\Lambda(\Lambda)^n}{d(\Lambda)} \leq \Gamma(\frac{n}{2} + 1)^2 \left(\frac{2}{\pi}\right)^n \cdot d(\Lambda)^2 \]

\[ \Rightarrow \chi_\Lambda(\Lambda)^n \leq 2^n \chi_n^{-1} d(\Lambda) = \Gamma(\frac{n}{2} + 1) \cdot 2^n \cdot \text{Vol}(\Lambda) \]

In other words, the ball

\[ B := \{ \|x\| : 2 \chi_n^{-1/2} d(\Lambda)^{1/n} \leq \|x\| \in \mathbb{R}^n \} \]

contains a non-zero lattice point.

We have

\[ \text{vol}(B) = \chi_n \cdot (2 \chi_n^{-1/2} d(\Lambda)^{1/n})^n = 2^n d(\Lambda) \]

There's a generalization of this observation called Minkowski's first theorem (or convex body theorem).