Lecture 6, 13.11.

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a) Modules do not have to be free!

b) In a free module, a generating set does not necessarily contain a basis

c) submodules of free modules do not have to be free

Ex: 3.51

a) Consider $\mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$-module.

If free, it would contain a copy of $\mathbb{Z}$ $\lightning$

b) Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. It is free.

$\{2,3\}$ is a generating set since $\underline{1} = -2+3$. But does not contain a basis

c) Let $R = \mathbb{Z}/6\mathbb{Z}$. Then R is a free R-module.

$U = 2R = \{0,2,4\} \subset R$ is a submodule.

But it is not free: otherwise, it would contain a copy of $R \Rightarrow |U| \geq 6$ $\lightning$

Still, some familiar properties do hold.

Lemma 3.52: Suppose V is free with basis $\{v_i\}_{i\in I}$. Let W be an R-module and let $\{w_i\}_{i\in I}$ be elements of W. Then $v_i \mapsto w_i$ extends to a morphism $V \to W$.

Proof: Straightforward. $\square$

Lemma 3.53: Let $f: V \to W$ be a morphism such that $\operatorname{Im} f$ is free.

Then $V \simeq \operatorname{Ker} f \oplus \operatorname{Im} f$.

Proof: Let $\{w_i\}_{i\in I}$ be a basis of $\operatorname{Im} f$. For each i choose $v_i \in f^{-1}(w_i)$. Since $\operatorname{Im} f$ free, can define a morphism

$s: \operatorname{Im} f \to V,\ w_i \mapsto v_i$.

Have $f \circ s = \mathrm{id}_{\operatorname{Im} f}$

Claim: $V = \ker f \oplus \operatorname{Im} s$

Let $v \in V$. Then $v = \underbrace{v - sf(v)}_{\in \ker f} + \underbrace{sf(v)}_{\in \operatorname{Im} s}$

$f(v - sf(v))$
$= f(v) - fsf(v)$
$= f(v) - f(v) = 0$

$\to V = \ker f + \operatorname{Im} s$

Let $v \in \ker f \cap \operatorname{Im} s \Rightarrow v = s(w)$ for some w

$\Rightarrow 0 = f(v) = fs(w) = w$

$\to V = \ker f \oplus \operatorname{Im} s.$ $\square$

<u>Lemma</u> 3.54: Let $R \neq 0$ and let V be a free R-module.
Then all bases of V have the same cardinality ($:=$ <u>dimension</u> $\dim_R V$ or <u>rank</u> $\operatorname{rk}_R V$ of V)

<u>Proof</u>:

Let M be a maximal ideal of R (exists by Prop 3.44, Remark 3.45).

$\rightsquigarrow k := R/M$ is a field.

Since M is an ideal, $MV = \{\sum m_i v_i ; m_i \in M, v_i \in V\}$ is a submodule of V

$\overline{V} := V/MV$ is an $R/M = k$-module

If $\{v_i\}_{i\in I}$ is a basis of V then $\{\overline{v}_i\}_{i \in I}$ is a k-basis of $\overline{V}$: $\square$

- generates: ✓
- linearly independent: $0 = \sum_i \overline{r}_i \overline{v}_i = \overline{\sum_i r_i v_i} \Rightarrow \sum_i r_i v_i \in MV$

$\Rightarrow \sum_i r_i v_i = \sum_i m_i v_i$ with $m_i \in M$

$\Rightarrow r_i = m_i \ \forall i$ since $\{v_i\}_{i\in I}$ a basis

$\Rightarrow \overline{r}_i = 0 \ \forall i$

$\Rightarrow |I| = \dim_k \overline{V}$, independent of basis $\square$

<u>Remark</u> 3.55: Well-definedness of dimension can fail for non-commutative rings!

<u>Lemma</u> 3.56: If V is finitely generated and free, it has a finite basis.

<u>Proof</u>: Let $\{v_i\}_{i\in I}$ be a basis. Let $\{v_1', \dots, v_n'\}$ be a finite generating set.

$v_j' = \sum_i r_{ij} v_i$

Let $I' := \{v_i \mid r_{ij} \neq 0 \text{ for some } j\}$, a finite set.

Then $v_j' \in R\cdot\{v_i\}_{i\in I'}$ $\forall j \Rightarrow V = R\cdot\{v_i\}_{i\in I'}$

Since $\{v_i\}_{i\in I'}$ is linearly independent, it is a basis. $\square$

How can we prove freeness?

Here is <u>an</u> obstruction to being free:

<u>Def</u> 3.57: A <u>torsion element</u> in V is an element $v\in V$ such that $rv=0$ for some non-zero divisor $r\in R$.

The set $T(V)$ of torsion elements is a submodule of V, called <u>torsion submodule</u>. V is called <u>torsion-free</u> if $T(V)=0$.

<u>Lemma</u> 3.58: If V is free, then $T(V)=0$.

<u>Proof</u>: Let $\{v_i\}_{i\in I}$ be a basis. Suppose there is $v\in V$ with $rv=0$ for some non-zero divisor r. We have $v = \sum_i r_i v_i$, $r_i\in R$. Hence

$0 = rv = \sum_i r r_i v_i \Rightarrow r r_i = 0$ $\forall i$. Since r is non-zero divisor

$\Rightarrow r_i = 0$ $\forall i \Rightarrow v = 0$ $\square$

<u>Lemma</u> 3.59: $V/T(V)$ is torsion-free.

Proof: Straightforward. $\square$

<u>Ex</u>[3.60]: $R = \mathbb{Z}/6\mathbb{Z}$ and $U = 2R = \{0,2,4\} \subset R$.

Then U is torsion-free but not free by Example 3.51

Torsion-free modules can be upgraded to vector spaces.

In all of the following let

R be an integral domain, $K := Q(R)$, V an R-module (not necessarily t.f for now)

Let $KV := \{(v,r) \mid r \in R\setminus\{0\}, v \in V\}/\sim$ where

$(v,r) \sim (v',r')$ iff there is $r'' \in R\setminus\{0\}$ s.th $r''(rv' - r'v) = 0$

Write $\frac{v}{r} = \frac{1}{r}v$ for (v,r). Then KV is a K-vector space with the obvious $+$ and K-action.

Note similarity to definition of $Q(R)$!

Why is there the additional r'' in the definition of $\sim$?

Why not say $\frac{v}{r} = \frac{v'}{r'}$ iff $vr' = v'r$?

Because of torsion! Suppose $0 \neq v$ a torsion element, i.e. $rv = 0$ for some $r \neq 0$.

If we would say $\frac{v}{1} = \frac{0}{1} \Leftrightarrow v \cdot 1 = 1 \cdot 0 \Leftrightarrow v = 0$, would have $\frac{v}{1} \neq 0$.

But look: $\frac{v}{1} = \frac{r}{r} \cdot \frac{v}{1} = \frac{rv}{r} = \frac{0}{r} = 0$ ↯

In the definition of $Q(R)$ we could drop this since R is a torsion-free R-module if R is an integral domain!

We see:

<u>Lemma</u>[3.61]: The kernel of $V \to KV$, $v \mapsto \frac{v}{1}$, is $T(V)$. $\square$

<u>Corollary</u>[3.62]: If V is torsion-free, then $V \to KV$ is injective. $\square$

<u>Remark</u>[3.63]: If V is torsion-free and $U \leq V$, then $KU \subseteq KV$.

However, if $U \subsetneq V$ it can happen that $KU = KV$.

Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module, and $2\mathbb{Z} \subsetneq \mathbb{Z}$

$\rightsquigarrow \quad \mathbb{Q}\mathbb{Z} = \mathbb{Q}$

$\qquad \mathbb{Q}(2\mathbb{Z}) = \mathbb{Q}$

<u>Lemma</u>[3.64]: If V is free, then $\dim_R V = \dim_K KV$.

<u>Proof</u>:

If $\{v_i\}_{i\in I}$ is an R-basis, then $\{v_i\}_{i\in I}$ generates KV as a K-space.

Moreover, $\sum_i \frac{r_i}{r_i'} v_i = 0 \Rightarrow$ multiply by $r' := \prod_i r_i' \neq 0$ to get $\sum_i r_i' v_i = 0$, $r_i' = \frac{r' r_i}{r_i'} \in R$

$\Rightarrow r_i' = 0 \ \forall i \Rightarrow r' r_i = 0 \ \forall i \Rightarrow r_i = 0 \ \forall i$ since $r' \neq 0$ and R integral domain. $\square$

<u>Lemma</u>[3.65]: R an integral domain, V f.g. R-module.

Then V is torsion-free iff V is a submodule of a free R-module.

<u>Proof</u>: A submodule of a free module is clearly torsion-free.

Conversely suppose that V is torsion-free.

Since V finitely generated $\Rightarrow KV$ is a finite-dimensional K-vector space.

Let $v_1', \dots, v_n'$ be a basis of KV.

Let $v_1, \dots, v_m$ be generators of V.

Can view $V \subseteq KV$ by Cor 3.62

$$\rightsquigarrow v_i = \sum_j \frac{r_{ij}}{r_{ij}'} v_j', \quad r_{ij}, r_{ij}' \in R$$

Let $r' := \prod_{i,j} r_{ij}' \neq 0$. Then $\frac{v_1'}{r'}, \dots, \frac{v_n'}{r'}$ is a basis of KV,

in particular linear independent over K, thus over R.

$\rightsquigarrow V' := R \cdot \{\frac{v_1'}{r'}, \dots, \frac{v_n'}{r'}\}$ is a free R-module in KV

Have $v_i \in V' \ \forall i \Rightarrow V \subseteq V'$. $\square$

Thm 3.66: Suppose R is a principal ideal domain. Then every finitely generated torsion-free R-module is already free

Proof: By Lemma 3.65 have $V \subseteq R^n$ for some $n \in \mathbb{N}$.

Do induction on n.

$n=1$: V is an ideal in R

Since R is a PID $\leadsto V = (r)$ for some $r \in R$.

Clearly, $\{r\}$ is a basis, so V is free.

$n>1$: Let $\pi : R^n \longrightarrow R^{n-1}$ be the projection onto the last $n-1$ summands.

Let $V' := \pi(V) \subseteq R^{n-1}$. Free by induction.

$\Longrightarrow V \simeq V' \oplus \operatorname{Ker}(\pi|_V)$ by Lemma 3.53.

Have $\operatorname{Ker}(\pi|_V) \subset R$ (first summand of R^n)

$\leadsto \operatorname{Ker}(\pi|_V)$ free by induction.

$\Rightarrow V$ free.

$\square$

Lemma 3.67: Let R be a principal ideal domain and V a finitely generated R-module. Then $V = T(V) \oplus F$, where F is free.

Proof: $V/T(V)$ is torsion-free by Lemma 3.54. It is finitely generated since V is $\Rightarrow V/T(V)$ free by Theorem 3.66.

Now have a surjective map $V \longrightarrow V/T(V) =: F$

$\leadsto V \simeq T(V) \oplus F$ by Lemma 3.53

$\square$

Question: Can we finally do some number theory again?

Answer: Alright!