Algorithmic Number Theory
WS 19/20

Prof. Dr. Ulrich Thiel
University of Kaiserslautern

Exercise Sheet 6

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Exercise 1. Show that the following sets are naturally in bijection:

(a) Lattices in $\mathbb{R}^n$ up to isomorphism.
(b) Matrices $A \in \mathrm{GL}_n(\mathbb{R})$ up to the relation $A \sim A'$ if $A' = PAT$ for some $P \in \mathrm{GL}_n(\mathbb{Z})$ and $T \in \mathrm{O}_n(\mathbb{R})$.
(c) Symmetric positive definite matrices $Q \in \mathrm{Mat}_n(\mathbb{R})$ up to the relation $Q \sim Q'$ if $Q' = PQP^t$ for some $P \in \mathrm{GL}_n(\mathbb{Z})$.

Exercise 2. Develop a formal algorithm to compute the quadratic supplement of a positive definite matrix.

Exercise 3. Let $Q \in \mathrm{Mat}_n(\mathbb{R})$ be symmetric and positive definite. Let $\tilde{Q}$ be the quadratic supplement of Q. Define $A \in \mathrm{Mat}_n(\mathbb{R})$ via $A_{ii} = \sqrt{\tilde{Q}_{ii}}$ and $A_{ij} \coloneqq \sqrt{\tilde{Q}_{ii}} \cdot \tilde{Q}_{ji}$ for $i \neq j$. Show that $Q = AA^t$.

Exercise 4. Let Λ be the lattice in $\mathbb{R}^3$ with Gram matrix

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \in \mathrm{Mat}_3(\mathbb{R})$$

with respect to the standard basis. Find all lattice vectors $x \in \Lambda$ with $\|x\|^2 \leq 3$.

Exercise 5. Determine the Minkowski lattice for the maximal order in the following number fields: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\zeta_3)$. Explicitly compute the volume of the fundamental region.