Algorithmic Number Theory
WS 19/20

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Exercise Sheet 7

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Exercise 1. Let L be a number field.

(a) For $\lambda \in \mathbb{R}_{>0}$ let

$$E_\lambda := \{x \in \mathbb{R}^{r+2s} \mid |x_1| + \ldots + |x_r| + \sqrt{2}\sqrt{x_{r+1}^2 + x_{r+2}^2} + \ldots + \sqrt{2}\sqrt{x_{r+2s-1}^2 + x_{r+2s}^2} \le \lambda\} .$$

Show that $\mathrm{vol}(E_\lambda) = \frac{2^r \pi^s}{n!} \lambda^n$.

(b) Show that if $j(\omega) \in E_\lambda$, then $\sum_{i=1}^n |\sigma_i(\omega)| \le \lambda$ and $|N_{L|\mathbb{Q}}(\omega)| \le n^{-n} \lambda^{-n}$.[1]

(c) Deduce that for any non-zero ideal I of $\mathcal{O}_L$ there is a non-zero $\omega \in I$ with

$$|N_{L|\mathbb{Q}}(\omega)| \le M_L [\mathcal{O}_L : I] ,$$

where

$$M_L := \frac{n!}{n^n} \left(\frac{4}{\pi}\right)^s \sqrt{|d_L|} .$$

(d) Deduce that

$$|d_L| \ge \left(\frac{n^n}{n!} \left(\frac{\pi}{4}\right)^s \right)^2$$

and that

$$|d_L| > 1 .$$

Exercise 2. Determine the density of the Minkowski lattice of the maximal order of the number fields $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\zeta_3)$ and compare this to the maximal density of lattices in $\mathbb{R}^2$.

Exercise 3. Consider the lattice $\Lambda \subset \mathbb{R}^5$ with basis given by the rows of the matrix

$$A := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(a) Find all lattice points $x \in \Lambda$ with $||x||^2 \le 4$.

(b) Determine the successive minima of Λ.

(c) Show that Λ has no basis $b_1, \ldots, b_5$ with $||b_i|| = \lambda_i(\Lambda)$ for all $i = 1, \ldots, 5$.

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[1] For the second inequality use the inequality between geometric and arithmetic mean.