Commutative Algebra

Exercise Sheet 1

Due date: 3 November 2020, 9:00 am. To be handed in digitally in OLAT.[1]

Exercise 1. An **idempotent** in a ring A is an element $e \in A$ with $e^2 = e$. Show the following:

(a) $Ae = \{ae \mid a \in A\} \subseteq A$ is a ring with the addition and multiplication from A. But it is not a subring unless $e = 1$.

(b) $1 - e$ is an idempotent as well.

(c) As a ring, A is isomorphic to the product $Ae \times A(1-e)$.

Exercise 2 (Chinese Remainder Theorem)**.** Let A be a ring and let $I_1, \ldots, I_n$ be ideals in A. Prove the following:

(a) The quotient maps $q_i : A \to A/I_i$ taken together induce a ring morphism

$$\varphi : A \to \prod_{i=1}^{n} A/I_i \ .$$

(b) φ is injective if and only if $\bigcap_{i=1}^{n} I_i = 0$.

(c) φ is surjective if and only if the I_i are mutually **coprime**, i.e. $I_i + I_j = A$ for all $i \neq j$.

(d) If the I_i are mutually coprime, then φ induces a ring isomorphism

$$A/\bigcap_{i=1}^{n} I_i \cong \prod_{i=1}^{n} A/I_i \ .$$

Moreover,

$$\bigcap_{i=1}^{n} I_i = \prod_{i=1}^{n} I_i \ .$$

Exercise 3. Let A be a ring. An element $x \in A$ is called **nilpotent** if there is $n \in \mathbb{N}$ with $x^n = 0$. Show the following:

(a) If $u \in A$ is a unit and $x \in A$ is nilpotent, then $u + x$ is a unit.

(b) If $x, y \in A$ are nilpotent, so is $x + y$.

Exercise 4. (a) Let R be a ring. A polynomial

$$f = r_0 + r_1 X + \cdots + r_n X^n \in R[X]$$

is a unit if and only if r_0 is a unit and all $r_1, \ldots, r_n$ are nilpotent. Exercise 3 might be helpful for this.

(b) Determine the units in $\mathbb{Z}[X]$.

[1] Or if OLAT does not work as e-mail at `schmitt@mathematik.uni-kl.de`.