Winter Term 20/21 | *Prof. Dr. Ulrich Thiel*
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Commutative Algebra

Exercise Sheet 10

Due date: 19 January 2021, 9:00 am.

Exercise 1. Explain the visualization of the spectrum of $\mathbb{Z}[X]$ in Figure 8.2 of the lecture notes.
Since this exercise does not have a well-defined correct solution, it is ***not*** *mandatory, but you can get 4 extra points for it.*
It might help to think about the following questions:

(i) The prime ideals of $\mathbb{Z}[X]$ are classified in Theorem 8.1.1, where they split into three different classes. To which classes do the prime ideals depicted in the drawing belong?

(ii) Theorem 8.1.1 also contains a statement about the possible inclusions between prime ideals. How are those represented?

(iii) What are those "doodles"? Why is the "bat-shaped" one in the upper right corner bigger than the others? (This is related to the question in (ii)).

(iv) Why is the line coming out of the (X^2+1)-doodle shaped like this? What is the difference between the small black dots and the larger black circles at the intersection points?

Exercise 2. (a) Show that if $A \subseteq B$ is an integral ring extension, then $\dim A = \dim B$.

(b) Let K be a field. Determine the dimension of the ring $K[X_1, X_2]/(X_2^2 - X_1^2 - X_1^3)$ (see Exercise 8.1).

Exercise 3. Determine the dimension of the following rings:

(a) $K[X_1, X_2, X_3]/(X_1X_2 - X_3^2)$

(b) $\mathbb{Z}_{(2)}[X]/(2X-1)$

Exercise 4. Generalizing the notion of (Krull) dimension of a ring, we define the (Krull) dimension $\dim(X)$ of a topological space X as the supremum of lengths of chains of irreducible closed subsets of X. Prove the following.

(a) $\dim(A) = \dim(\operatorname{Spec}(A))$ for a ring A.

(b) If $Y \subseteq X$, then $\dim Y \leq \dim X$.

(c) If X is irreducible and $\dim X < \infty$, then $Y \subsetneq X$ with Y closed implies $\dim Y < \dim X$.

(d) $\dim X = \sup_\lambda \dim X_\lambda$, where the X_λ are the irreducible components of X.