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Commutative Algebra

Exercise Sheet 11

Due date: 26 January 2021, 9:00 am.

Exercise 1. Find an example of a ring which is not noetherian but of finite dimension.

Exercise 2. Let K be a field and let A be a K-algebra which is also an integral domain.

(a) Show that $\dim A \leq \operatorname{trdeg}_K A$.

(b) Find an example where the inequality in (a) is strict.

Exercise 3. Let p be a prime number and let $A := \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ in the prime ideal (p). Show the following:

(a) $M_1 := (pX - 1)$ and $M_2 := (p, X)$ are maximal ideals in $A[X]$.

(b) $\operatorname{codim}(M_1, A[X]) = 1$ and $\operatorname{codim}(M_2, A[X]) = 2$.

(c) $\dim(M_1) + \operatorname{codim}(M_1, A[X]) < \dim(A[X])$.

(d) $A[X]$ is catenary.

Exercise 4. Let K be a field and let $A := K[X, Y]/(X^2, XY)$. We denote by $\overline{X}$ and $\overline{Y}$ the image of X and Y, respectively, in A. Show the following:

(a) $\dim A = 1$.

(b) $(\overline{X})$ is the unique minimal prime ideal of A.

(c) $\operatorname{codim}\big((\overline{Y}), A\big) = 1$.