Winter Term 20/21

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Commutative Algebra

Exercise Sheet 12

Due date: 2 February 2021, 9:00 am.

Exercise 1. Let K be a field and let $A := K[X_1, X_2]/(X_1^2 X_2 - 1)$. Find a Noether normalization of A, that is, find an algebraically independent subset $\{Y_1, \dots, Y_d\} \subseteq A$ over K such that the extension $K[Y_1, \dots, Y_d] \subseteq A$ is finite.
Hint: There are at least two ways to approach this: you can follow the construction in the proof of Theorem 8.3.9, but looking at a plot of the vanishing set $X_1^2 X_2 - 1 = 0$ might also help.

Exercise 2. Let A be a noetherian local ring and let $x_1, \dots, x_c$ be a regular sequence. Show that any permutation of $x_1, \dots, x_c$ is a regular sequence.

Exercise 3. Let K be a field.

(a) Show that $K[X_1, \dots, X_n]$ is regular in the origin $(X_1, \dots, X_n)$, that is, that the localization $K[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ is a regular local ring.

(b) Show that if A is a noetherian regular local ring with maximal ideal M and $0 \neq x \in M^2$, then $A/(x)$ is not regular.

Exercise 4. Let K be a field of characteristic $\neq 2, 3$ and let $A := K[X_1, X_2]/(X_1^3 - X_2^2)$.

(a) Show that A is regular in $M := (X_1 - 1, X_2 - 1)$.

Hint: Prove that MA_M is principal.

(b) Show that A is not regular in the origin $M := (X_1, X_2)$.

Hint: Exercise 3.