Commutative Algebra

Exercise Sheet 13

Due date: 9 February 2021, 9:00 am.

Exercises 1 and 2 are "leftovers"; you can find them in sections 6.1 and 7.3 of the lecture notes.

Exercise 1. A subset Y of a topological space X is called **very dense** if the map $U \mapsto U \cap Y$ is a bijection between the open subsets of X and the open subsets of Y.[(1)]

(a) Show that a subset Y of a topological space X is very dense if and only if $\overline{Z \cap Y} = Z$ for any closed $Z \subseteq X$.

(b) Show that a ring A is Jacobson if and only if the subset $\mathrm{Max}(A)$ is very dense in $\mathrm{Spec}(A)$.

Hint: Use (a) (obviously) and the properties of V and I.

Exercise 2. A topological space X is called **noetherian** if the partially ordered set of open subsets of X is noetherian. This is obviously equivalent to the partially ordered set of *closed* subsets being *artinian*. Prove the following:

(a) A subspace of a noetherian space is noetherian as well.

(b) A noetherian space has only finitely many irreducible components.

Hint: we already know that irreducible components are closed subsets. Consider the set Σ of all closed subsets not having finitely many irreducible components. The assumption $\Sigma \neq \emptyset$ leads to a contradiction after choosing a minimal element of Σ (why does this exist?).

(c) If A is a noetherian ring, then $\mathrm{Spec}(A)$ is a noetherian space.

(d) A noetherian ring has only finitely many minimal prime ideals.

Exercise 3 (8 points). Let $v : K^\times \to \mathbb{Z}$ be a discrete valuation on a field K and let A be the corresponding valuation ring. Show the following:[(2)]

(a) Check that A is indeed a ring, so that $1 \in A$ and that A is closed under addition and multiplication.

(b) $A^\times = \{x \in K \mid v(x) = 0\}$.

(c) A is local with maximal ideal $M := \{x \in A \mid v(x) \geq 1\}$.

[(1)] More generally, one calls a morphism $f : Y \to X$ of topological spaces a **quasi-homeomorphism** if the map $U \mapsto f^{-1}(U)$ is a bijection between the open subsets of X and the open subsets of Y. A subset Y of a topological space X is hence very dense if the inclusion $Y \to X$ is a quasi-homeomorphism.

[(2)] Beware of circular arguments! You should not use any results from the lecture notes showing up after Exercise 9.1.15.

(d) An element $\pi \in A$ with $v(\pi) = 1$ is called a **uniformizer**. Fixing such an element, every $x \in A$ can be uniquely written as $x = u\pi^n$ with $u \in A^\times$ and $n \in \mathbb{N}$.

(e) For $x \in A$ the valuation $v(x)$ is the largest number n such that π^n divides x.

(f) $M^n = (\pi^n) = \{x \in A \mid v(x) \geq n\}$ and every non-zero ideal in A is of this form.

(g) If $x \in K$, then $x \in A$ or $x^{-1} \in A$.

(h) The fraction field of A is equal to K.