Winter Term 20/21 *Prof. Dr. Ulrich Thiel*
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Commutative Algebra

Exercise Sheet 2

Due date: 10 November 2020, 9:00 am.

Exercise 1. Let R be a ring.

(a) Let Λ be a set and let $\Lambda' \subseteq \Lambda$ be a subset. There is a canonical R-algebra isomorphism

$$R[(X_\lambda)_{\lambda\in\Lambda}] \cong \big(R[(X_\lambda)_{\lambda\in\Lambda\setminus\Lambda'}]\big)[(X_\lambda)_{\lambda\in\Lambda'}]\,.$$

Comment: This is as easy as it looks.

(b) The ring R is an integral domain if and only if the polynomial ring $R[X_1,\dots,X_n]$ in finitely many variables is an integral domain.

(c) The polynomial ring $R[X]$ is a principal ideal domain if and only if R is a field.

Exercise 2. Let K be a field.

(a) The ideal $(X_1,\dots,X_r)$ is a prime ideal in $K[X_1,\dots,X_n]$ for each $1 \le r \le n$.

(b) Let $p \in K[X]$ and let $\varphi : K[X_1,X_2] \to K[X]$ be the morphism defined by $X_1 \mapsto X$ and $X_2 \mapsto p$, i.e. $\varphi(f) = f(X,p)$. Show that $\ker\varphi = (X_2 - p(X_1))$ and conclude that this is a prime ideal in $K[X_1,X_2]$.

(c) Show that $(X_1 + X_2 - 1) \subseteq K[X_1,X_2]$ is a prime ideal.

Exercise 3. (a) A ring A has a unique maximal ideal if and only if $A \setminus A^\times$ is an ideal in A.

(b) Find an example of a ring as in (a), where $A \setminus A^\times$ is not $\{0\}$, i.e. which is not a field.

Exercise 4. (a) Any prime ideal $P \neq 0$ in a principal ideal domain is maximal.

(b) Let K be a field. Find a prime ideal in $K[X,Y]$ which is *not* maximal. Do the same for $\mathbb{Z}[X]$.