Commutative Algebra

Exercise Sheet 3

Due date: 17 November 2020, 9:00 am.

Exercise 1. (a) Show that a map $f : X \to Y$ between topological spaces is continuous if and only if for any open subset $V \subseteq Y$ the preimage $f^{-1}(V) \subseteq X$ is open.

(b) Let A be a ring and let $I \trianglelefteq A$ be an ideal. The topological spaces $\mathrm{Spec}(A/I)$ and $\mathrm{V}(I)$ are homeomorphic, i.e. isomorphic as topological spaces.

Exercise 2. Let A be a ring.

(a) The sets

$$\mathrm{D}(f) := \mathrm{Spec}(A) \setminus \mathrm{V}(f)$$

define a basis for the Zariski topology on $\mathrm{Spec}(A)$, i.e. every open subset is a union of such sets.

(b) The topological space $\mathrm{Spec}(A)$ is quasi-compact, i.e. every open cover of $\mathrm{Spec}(A)$ has a finite subcover.

Exercise 3. Let A be a ring.

(a) Show that $\sqrt{I^n} = \sqrt{I}$ for any $I \trianglelefteq A$.

(b) Show that $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$ for any $I, J \trianglelefteq A$.

(c) Find an example where $\sqrt{I + J} \neq \sqrt{I} + \sqrt{J}$.

Exercise 4. Let K be a field. Determine the following radicals.

(a) $\sqrt{(X^3 - X^2 - X + 1)}$ in $K[X]$.

(b) $\sqrt{(X_1^2 - X_2X_3, X_1(1 - X_3))}$ in $K[X_1, X_2, X_3]$.

Hint: Use Corollary 2.6.5.