Commutative Algebra

Exercise Sheet 4

Due date: 24 November 2020, 9:00 am.

Exercise 1. Let A be a ring.

(a) For a family $(V_\lambda)_{\lambda \in \Lambda}$ of A-modules and an A-module W there is a canonical isomorphism

$$\mathrm{Hom}_A\Big(\bigoplus_{\lambda \in \Lambda} V_\lambda, W\Big) \simeq \prod_{\lambda \in \Lambda} \mathrm{Hom}_A(V_\lambda, W)$$

of A-modules.

(b) For an A-module V and a family $(W_\lambda)_{\lambda \in \Lambda}$ of A-modules there is a canonical isomorphism

$$\mathrm{Hom}_A\Big(V, \prod_{\lambda \in \Lambda} W_\lambda\Big) \simeq \prod_{\lambda \in \Lambda} \mathrm{Hom}_A(V, W_\lambda)$$

of A-modules.

(c) Let $(V_\lambda)_{\lambda \in \Lambda}$ be a family of A-modules and for each λ let $U_\lambda \subseteq V_\lambda$ be a submodule. Then $\bigoplus_{\lambda \in \Lambda} U_\lambda$ is naturally a submodule of $\bigoplus_{\lambda \in \Lambda} V_\lambda$ and there is a canonical A-module isomorphism

$$\Big(\bigoplus_{\lambda \in \Lambda} V_\lambda\Big) / \Big(\bigoplus_{\lambda \in \Lambda} U_\lambda\Big) \simeq \bigoplus_{\lambda \in \Lambda} V_\lambda / U_\lambda \ .$$

Exercise 2. Let A be a non-zero ring such that every ideal is a free A-module. Then A is a principal ideal domain.

Exercise 3. (a) Let $\varphi : A \to B$ be a ring morphism and let V be an A-module. If V is free with basis $(v_\lambda)_{\lambda \in \Lambda}$, then V^B is a free B-module with basis $(1 \otimes v_\lambda)_{\lambda \in \Lambda}$.

(b) Let I be an ideal in a ring A and let V be an A-module. Then there is a canonical isomorphism

$$V/IV \simeq (A/I) \otimes_A V$$

of (A/I)-modules, where the scalar extension is taken with respect to the quotient map $A \to A/I$.

Exercise 4. (a) Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}$ with $d := \gcd(m, n)$.

(b) Let V be a $\mathbb{Z}$-module with $\mathrm{T}(V) = V$, e.g. a finite abelian group or $\mathbb{Q}/\mathbb{Z}$. Then $\mathbb{Q} \otimes_{\mathbb{Z}} V = 0$.