Winter Term 20/21 *Prof. Dr. Ulrich Thiel*
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Commutative Algebra

Exercise Sheet 5

Due date: 1 December 2020, 9:00 am.

Exercise 1. (a) Show that for any two submodules U, U' of an A-module V there is a canonical short exact sequence

$$0 \longrightarrow U \cap U' \longrightarrow U \oplus U' \longrightarrow U + U' \longrightarrow 0 \ .$$

(b) The cokernel of an A-module morphism $f : V \to W$ is defined as

$$\operatorname{Coker} f := W/\operatorname{Im} f \ .$$

Show that there is a canonical exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow V \xrightarrow{f} W \longrightarrow \operatorname{Coker} f \longrightarrow 0 \ .$$

Exercise 2. (a) Show that the functor $\operatorname{Hom}_A(V, -) : A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ is left-exact.

(b) Show that the functor $\operatorname{Hom}_A(V, -) : A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ is in general not exact. *Hint:* Consider $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ and the quotient map $q : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$.

Exercise 3. Prove that the following are equivalent for an A-module P:

(a) P is projective.

(b) P is a direct summand of a free A-module, i.e. there is an A-module Q such that $P \oplus Q \simeq A^{(\Lambda)}$ for some Λ.

(c) Every short exact sequence of the form

$$0 \longrightarrow V' \longrightarrow V \xrightarrow{g} P \longrightarrow 0$$

splits, i.e. there is an A-module morphism $s : P \to V$ such that $g \circ s = \mathrm{id}_P$ (such an s is called a section of g).[1]

(d) For every morphism $h : P \to W$ and every surjective morphism $f : V \to W$ there is a morphism $\tilde{h} : P \to V$ such that the diagram

$$\begin{tikzcd} & V \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"'] & W \end{tikzcd}$$

commutes.

[1] You may use the following fact without proving it: a short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

splits if and only if $V \simeq V' \oplus V''$.

Exercise 4. (a) Show that projective modules are flat.

(b) Show that there are projective modules which are not free.

Hint: Let $A := \mathbb{Z}/6\mathbb{Z}$ and consider the A-module $\mathbb{Z}/2\mathbb{Z}$.

(c) Show that there are flat modules which are not projective.[2]

Hint: Consider the $\mathbb{Z}$-module $\mathbb{Q}$.

[2] You may use the following fact without proving it: over a principal ideal domain submodules of free modules are free.