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Commutative Algebra

Exercise Sheet 7

Due date: 15 December 2020, 9:00 am.

Exercise 1 (Five lemma)**.** Consider a commutative diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, two heads, "l"] & W \arrow[r, "g"] \arrow[d, "m", "\simeq"'] & X \arrow[r, "h"] \arrow[d, "n"] & Y \arrow[r, "j"] \arrow[d, "p", "\simeq"'] & Z \arrow[d, hook, "q"] \\ V' \arrow[r, "f'"'] & W' \arrow[r, "g'"'] & X' \arrow[r, "h'"'] & Y' \arrow[r, "j'"'] & Z' \end{tikzcd}$$

of module morphisms, where

(i) the two rows are exact;

(ii) m and p are isomorphism;

(iii) l is surjective;

(iv) q is injective.

Show that n is an isomorphism.

Exercise 2. (a) Show that projectivity of finitely presented modules is a local property.

(b) Show that for a module V over an arbitrary ring A the following are equivalent:
  (i) V is finitely presented and flat.
  (ii) V is finitely generated and projective.

Exercise 3. The aim of this exercise is to construct a finitely generated flat module which is not projective. You can achieve a maximum of **8 points** instead of the usual 4 points with this exercise.

(a) Prove **Schanuel's Lemma**: if

$$0 \longrightarrow K \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

and

$$0 \longrightarrow K' \longrightarrow P' \xrightarrow{f'} M \longrightarrow 0$$

are short exact sequences of module morphisms over a ring A with P and P' projective, then there is an isomorphism

$$K' \oplus P \simeq K \oplus P' \ .$$

Hint: consider $X := \{(p, p') \in P \oplus P' \mid f(p) = f'(p')\}$ and show that $X \simeq K' \oplus P$ and $X \simeq K \oplus P'$.

(b) Show that if A is a ring and V is a finitely presented A-module, then the kernel $\operatorname{Ker} f$ of a surjective morphism $f : W \twoheadrightarrow V$ from a finitely generated A-module W is finitely generated as well.

Hint: since W is finitely generated, there is a surjective morphism $g : A^k \twoheadrightarrow W$. From this you get an exact sequence $0 \longrightarrow K' \longrightarrow A^k \xrightarrow{f \circ g} V \longrightarrow 0$. Moreover, we have an exact sequence $0 \longrightarrow K \longrightarrow A^n \longrightarrow V \longrightarrow 0$ with K finitely generated since V is finitely presented. Now, use Schanuel's Lemma and note that $g(\operatorname{Ker}(fg)) = \operatorname{Ker} f$.

(c) Consider the $\mathbb{Z}$-module $A_0 := \bigoplus_{n \in \mathbb{N}} (\mathbb{Z}/2\mathbb{Z})$. With respect to component-wise addition and multiplication this is a ring without unit. But $A := \mathbb{Z} \oplus A_0$ becomes a ring with unit $(1, 0)$ with respect to component-wise addition and the multiplication defined by $(n, a_0) \cdot (n', a_0') := (nn', na_0' + n'a_0 + a_0 a_0')$. (You are supposed to show the facts about the units.)

(d) Let $a := (2, 0) \in A$ and $V := (a) \trianglelefteq A$. This is a finitely generated A-module. We have a short exact sequence $0 \longrightarrow \operatorname{Ann}_A(a) \longrightarrow A \xrightarrow{\varphi} V \longrightarrow 0$, where φ is multiplication by a. Show that the ideal $\operatorname{Ann}_A(V) \trianglelefteq A$ is not finitely generated and conclude that V is not finitely presented.

(e) Show that V is not projective.

Hint: can a finitely generated but not finitely presented module be projective?

(f) Show that V is flat.

Hint: show that V is locally flat, i. e. V_P is flat for all $P \in \operatorname{Spec} A$. Treat the cases $A_0 \not\subseteq P$ and $A_0 \subseteq P$ separately.