Commutative Algebra

Exercise Sheet 8

Due date: 5 January 2021, 9:00 am.

Exercise 1. Let K be a field. In this exercise, you will compute the normalization of $A := K[X_1, X_2]/(X_2^2 - X_1^2 - X_1^3)$. You may assume that A is an integral domain without proving it.(1)

(a) Let x_i be the image of $X_i \in K[X_1, X_2]$ in A. Show that $\frac{x_2}{x_1} \in Q(A)$ is integral over A.

(b) Show that $A \subseteq K[\frac{x_2}{x_1}] \subseteq Q(A)$.

Hint: can you express x_1 and x_2 in terms of $\frac{x_2}{x_1}$?

(c) Show that $K[\frac{x_2}{x_1}]$ is the normalization of A.

Hint: we have a surjective map $\varphi : K[t] \to K[\frac{x_2}{x_1}]$. Hence, $K[\frac{x_2}{x_1}]$ is a principal ideal domain, hence?

(d) Show that $K[\frac{x_2}{x_1}]$ is (isomorphic to) a polynomial ring in one variable, i. e. the normalization of A is the affine line.

Hint: consider again the surjection $\varphi : K[t] \to K[\frac{x_2}{x_1}]$. Then $K[t]/\operatorname{Ker}\varphi \simeq K[\frac{x_2}{x_1}]$ and we want to show that $\operatorname{Ker}\varphi = 0$. If $\operatorname{Ker}\varphi$ were non-zero, then $\operatorname{Ker}\varphi$ would be a maximal ideal, hence?

Exercise 2. Show that being normal is a local property for integral domains.

Exercise 3. (a) Let $A \subseteq B$ be an integral extension. Show that if $Q \in \operatorname{Spec}(B)$ lies over $P \in \operatorname{Spec}(A)$, so $P = Q \cap A$, then Q is maximal if and only if P is maximal.

(b) Let $\varphi : A \to B$ be an integral ring morphism, i. e. the extension $\varphi(A) \subseteq B$ is integral (this generalizes integral extensions). Show that the morphism $\varphi^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is closed, i. e. φ^* maps closed subsets to closed subsets.

Exercise 4. In this exercise, you will show that finite morphisms have finite fibres.

(a) Show that if A is a finite-dimensional algebra over a field and A is also an integral domain, then A is already a field.

Hint: multiplication with an element from A defines a vector space endomorphism $A \to A$.

(1) This is essentially Exercise 5.2.6 of the lecture notes, but I trimmed it a bit to make it fit into the "4 points scheme". You are invited to draw a picture of the corresponding vanishing set, but I wouldn't know how to mark this.

Prof. Dr. Ulrich Thiel
Johannes Schmitt, MSc

(b) Show that if A is a finite-dimensional algebra over a field, then $\mathrm{Spec}(A)$ is finite and consists only of maximal ideals.

Hint: maximality of all prime ideals follows from the previous part. To prove finiteness note that if we have r maximal ideals, then $\dim_K A \geq r$ by the Chinese remainder theorem.

(c) Let $\varphi : A \to B$ be a finite morphism, i. e. B is a finitely generated A-module via φ (this generalizes finite extensions). Show that the fibres of $\varphi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are finite.