Winter Term 20/21

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Commutative Algebra

Exercise Sheet 9

Due date: 12 January 2021, 9:00 am.

Exercise 1. (a) Show that a finite direct sum of noetherian modules is noetherian.

(b) Show that for a finitely generated module V over a noetherian ring A the following are equivalent:

(i) V is flat;

(ii) V is projective;

(iii) V is locally free, i. e. V_P is a free A_P-module for all $P \in \operatorname{Spec}(A)$.

Exercise 2. (a) Show that if V is a noetherian A-module and $S \subseteq A$, then $S^{-1}V$ is a noetherian $S^{-1}A$-module.

(b) Find an example showing that the porperty of rings being noetherian is not a local property.

Hint: consider the ring $A := \prod_{i\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ and show that this ring is locally a field, i. e. A_P is a field for any $P \in \operatorname{Spec}(A)$. To this end, use the fact that $x^2 = x$ for all $x \in A$.

Exercise 3. Let p be a prime number and let $V_p := (\{p\}^{-1}\mathbb{Z})/\mathbb{Z}$. Then V_p is an artinian but not noetherian $\mathbb{Z}$-module.
Hint: Show that any proper submodule[(1)] of V_p is generated by $1/p^n$ for some $n \in \mathbb{N}$.

Exercise 4. (a) Show that the length ℓ of modules is an additive function, i. e. if $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is a short exact sequence, then

$$\ell(V) = \ell(V') + \ell(V'') \ .$$

(b) Show that an artinian ring is a finite direct sum of local artinian rings.

[(1)] Update 4 January: the first version just said “a submodule”.