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Exercise Sheet 1

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Exercises with numbers in brackets are taken from the book "An invitation to algebraic geometry" by Smith et. al. (2000).

Exercise 1 [1.1.2]**.** A *subvariety* of an affine algebraic variety $V \subset \mathbb{C}^n$ is an affine algebraic variety $W \subset \mathbb{C}^n$ that is contained in V.

(a) Show that the set $\mathbf{U}(n)$ of unitary $n \times n$ matrices is *not* an affine algebraic subvariety of $\mathbb{C}^{n^2}$. (Recall: A complex $n \times n$ matrix A is call unitary iff $(A^*)^T = A^{-1}$, where $*$ is complex conjugation and ^{T} transposition.)

(b) Show, however, that $\mathbf{U}(n)$ can be described as the zero locus of a collection of polynomials with real coefficients in $\mathbb{R}^{2n^2}$ that is, it is a *real algebraic variety*.

Exercise 2 [1.2.1]**.** Show that the union of two affine algebraic varieties in complex n-space is an affine algebraic variety.

Exercise 3 [1.2.2]**.** Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

Exercise 4 [1.2.3]**.**

(a) Show that the twisted cubic curve $\mathbb{V}\left(x^2 - y, x^3 - z\right)$ consists of all points in $\mathbb{A}^3$ of the form (t, t^2, t^3), where $t \in \mathbb{C}$.

(b) Show that the twisted cubic curve is not contained in any affine hyperplane in $\mathbb{A}^3$.