Exercise Sheet 2

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Exercises with numbers in brackets are taken from the book "An invitation to algebraic geometry" by Smith et. al. (2000).

Exercise 1 [2.3.1]. Show that under the $\mathbb{V}$ and $\mathbb{I}$ operators prime ideals correspond to irreducible algebraic varieties. Conclude that the irreducible components of an affine algebraic variety V correspond to the minimal prime ideals above $\mathbb{I}(V)$.

Exercise 2 . Let $F: V \to W$ be a morphism of affine algebraic varieties.

a. [2.5.1] Show that the pullback $F^\sharp: \mathbb{C}[W] \to \mathbb{C}[V]$ is injective if and only if $F(V)$ is dense in W. We then say that F is a *dominant* morphism.

b. [2.5.2] Show that the pullback $F^\sharp: \mathbb{C}[W] \to \mathbb{C}[V]$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W.

Exercise 3 [2.5.3]. If $F = (F_1, \dots, F_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism, then show that the Jacobian determinant

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

is a nonzero constant polynomial.
(Entertaining fact: It is not known whether the converse is true. This is a famous open problem known as the *Jacobian conjecture*.)

Exercise 4 [2.6.4]. Let $R \coloneqq \mathbb{C}[x, y]/(x^2)$. Show that $\operatorname{maxSpec}(R)$ is homeomorphic to $\mathbb{A}^1$. What is your conclusion?