Algebraic Geometry WS 2024/2025 Prof. Dr. Ulrich Thiel
RPTU Kaiserslautern–Landau

Exercise Sheet 3

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Exercises with numbers in brackets are taken from the book "An invitation to algebraic geometry" by Smith et. al. (2000).

Exercise 1. Let I be a homogeneous ideal in $\mathbb{C}[x_0, \dots, x_n]$.

1. Show that I is prime if and only if $fg \in I$ implies $f \in I$ or $g \in I$ for any *homogeneous* elements $f, g \in I$.
2. Show that $\sqrt{I}$ is homogeneous as well.

Exercise 2. The notions of *irreducible*, *irreducible components*, and *dimension* that we introduced for affine varieties work verbatim for general topological spaces. Consider $\mathbb{P}^n$ with the Zariski topology.

1. Let V be a non-empty projective variety in $\mathbb{P}^n$. Show that the following are equivalent: a) $V \subset \mathbb{P}^n$ is irreducible; b) the affine cone $C(V)$ is irreducible; c) $\mathbb{I}(V)$ is a prime ideal.
2. Show that $\mathbb{P}^n$ is irreducible.
3. Show that $\dim(\mathbb{P}^n) = n$. [Hint: Prove that, in general, the dimension can be determined from an open cover.]
4. Show that the closure in $\mathbb{P}^n$ of an irreducible affine variety is irreducible as well. [Hint: This, again, is a general topological fact.]

Exercise 3. Explicitly describe the projective closure in $\mathbb{P}^3$ of the twisted cubic curve: give both a parametric presentation and a presentation as a zero set of polynomials. [Hint: You do not have to be completely rigorous, use a limit argument as in class for the parabola.]

Exercise 4. Consider the affine variety $V = \mathbb{V}(y - x^2, z - xy) \subset \mathbb{A}^3$.

1. Show that V is the twisted cubic curve.
2. Show that $\mathbb{I}(V) = (y - x^2, z - xy)$.
3. Consider the *homogenizations* $wy - x^2$ and $wz - xy$ and let $W \subset \mathbb{P}^3$ be the projective variety they define. Show that W is *strictly larger* than the projective closure $\overline{V}$ of V.

Exercise 5 [3.3.2]. Consider the affine variety $V = \mathbb{V}(y - x^2, z^2 - 2xyz + y^3) \subset \mathbb{A}^3$.

1. Show that V is the twisted cubic curve.
2. Show that $\mathbb{I}(V) = (y - x^2, z - xy)$.
3. Consider the *homogenizations* $wy - x^2$ and $wz^2 - 2xyz + y^3$ and let $W \subset \mathbb{P}^3$ be the projective variety they define. Show that W is equal to the projective closure $\overline{V}$ of V but the ideal $(wy - x^2, wz^2 - 2xyz + y^3)$ is *not* radical.