Algebraic Geometry WS 2024/2025 Prof. Dr. Ulrich Thiel
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Exercise Sheet 7

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Exercise 1 [5.3.6].
Let X and Y be quasi-projective varieties and let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be the two natural projections. Show that the Segre product $X \times Y$ enjoys the following universal property of products:

> If Z is any quasi-projective variety admitting morphisms $p_1 \colon Z \to X$ and $p_2 \colon Z \to Y$, then there is a *unique* map $\mu \colon Z \to X \times Y$ such that the compositions $\pi_i \circ \mu$ agree with p_i.

Exercise 2 [5.3.3].
Let X and Y be two affine varieties.

a. Show that $X \times Y$ is the Cartesian product (i.e. simply the product in the category of affine varieties).

b. Show that $X \times Y$ is an affine variety.

c. Describe the coordinate ring of $X \times Y$.

Note that even in the affine case, our *definition* of product uses the Segre map, the affine varieties being thought of as quasi-projective varieties in some projective spaces.

Exercise 3 [5.3.4].
Show that the topology defined on the product above is *not* the product topology, except when one of the varieties is just a finite collection of points.

Exercise 4 [5.3.5].
Show that the subset of all degenerate plane conics naturally forms a closed subset in $\mathbb{P}^5$ of dimension four, isomorphic to the projection, from $\mathbb{P}^8$, of the Segree four-fold Σ_{22}, where Σ_{22} denotes the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under the Segre map.