Algebraic Geometry WS 2024/2025 Prof. Dr. Ulrich Thiel
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# Exercise Sheet 8

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Exercises with numbers in brackets are taken from the book "An invitation to algebraic geometry" by Smith et. al. (2000).

**Exercise 1** (5.4.1).
Fix an irreducible conic $C$ in $\mathbb{P}^2$. Show that the set of lines in $\mathbb{P}^2$ that fail to meet the conic in exactly two distinct points is a closed subvariety of the Grassmannian of all lines in $\mathbb{P}^2$, $\mathrm{Gr}\,(2,3)$.

**Exercise 2.**
Give explicit equations for $\mathrm{Gr}(2,4)$. The number of your equations should be minimal.

**Exercise 3** Plücker embedding via exterior power.

a. State the definition of the $k$-th exterior power $\wedge^k\mathbb{C}^n$. Use the reference of your choice.

b. State the definition of the projectivization $\mathbb{P}(\wedge^k\mathbb{C}^n)$ of the vector space $\wedge^k\mathbb{C}^n$. Again, use the reference of your choice.

c. Show that $\mathbb{P}(\wedge^k\mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{k}-1}$

d. Let $W = \mathrm{Span}\,(w_1, \ldots, w_k) \subset \mathbb{C}^n$ be a $k$-dimensional vector space and $\{e_1, \ldots, e_n\}$ a basis of $\mathbb{C}^n$. Let $w_j = \sum\limits_{i=1}^{n} a_{j,i} e_i$. Verify the following identity:

$$w_1 \wedge \cdots \wedge w_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \left( \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma) \cdot a_{1,i_{\sigma(1)}} \cdot \ldots \cdot a_{k,i_{\sigma(k)}} \right) \cdot e_{i_1} \wedge \cdots \wedge e_{i_k} \, .$$

In this expression, $S_k$ is the symmetric group on $k$ indices.

e. Describe the projective equivalence class $[w_1 \wedge \cdots \wedge w_k] \in \wedge^k\mathbb{C}^n$. (Hint: Look for minors.)

f. Conclude that the following map is well-defined and agrees with the Plücker map as introduced in the lecture via minors:

$$\iota \colon \mathrm{Gr}\,(k,n) \to \mathbb{P}\left(\wedge^k V\right) \, , \, W = \mathrm{Span}\,(w_1, \ldots, w_k) \mapsto [w_1 \wedge \cdots \wedge w_k] \, .$$

**Exercise 4** (5.5.1).
Show that a subvariety of $\mathbb{P}^n$ has degree one if and only if it is a linear subvariety.