Algebraic Geometry WS 2024/2025 Prof. Dr. Ulrich Thiel
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Exercise Sheet 10

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Exercises with numbers in brackets are taken from the book "An invitation to algebraic geometry" by Smith et. al. (2000).

Exercise 1. Prove that a generic hyperplane section of a smooth projective variety is smooth. [This is basically Bertini's theorem but in class we also said that this happens precisely on an open subset, which you do not need to prove here.]
[Hint: Consider the incidence correspondence $\{(p, H) \in V \times \check{\mathbb{P}}^n \mid \mathsf{T}_p V \subset H\}$.]

Exercise 2. Prove that the general hypersurface of degree d in $\mathbb{P}^n$ is smooth. [Hint: Use Bertini's theorem and the Veronese embedding.]

Exercise 3 [6.1.4]**.** Let $V \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous irreducible polynomial F. Show that $\mathrm{Sing}(V) = \mathbb{V}(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n})$.

Exercise 4 [6.1.2+6.1.3]**.** Prove that the following ways to define the projective tangent space $\mathsf{T}_p V \subset \mathbb{P}^n$ of a projective variety $V \subset \mathbb{P}^n$ at a point p are equivalent:

1. If $\mathbb{I}(V) = (F_1, \ldots, F_r)$ with F_i homogeneous, then

$$\mathsf{T}_p V = \mathbb{V}(\mathrm{d}F_1|_p(x-p), \ldots, \mathrm{d}F_r|_p(x-p)) \subset \mathbb{P}^n .$$

2. If $\tilde{p}$ is a representative of p on the affine cone $C(V) \subset \mathbb{A}^{n+1}$ of V, then

$$\mathsf{T}_p V = \mathbb{P}(\mathsf{T}_{\tilde{p}} C(V)) .$$

3. If $U_i \subset \mathbb{P}^n$ is an affine chart containing p, then

$$\mathsf{T}_p V = \overline{\mathsf{T}_p(U_i \cap V)} ,$$

   where we consider $U_i \cap V$ as an affine variety in $U_i \simeq \mathbb{A}^n$ and we take the projective closure of $\mathsf{T}_p(U_i \cap V)$.

Exercise 5 [6.2.1]**.** Find defining equations for the total tangent bundle of an affine variety $V \subset \mathbb{A}^n$ in terms of its defining equations. For a projective variety show that the total projective tangent bundle is a projective variety.