

Exercise Sheet 3

Release: November 9, 2023

Deadline: **November 16, 2023 by 10:00 a.m. Kaiserslautern**

Each exercise is worth 4 points. You need a minimum of 50% of the total points of all exercise sheets by the end of the semester in order to obtain the “Schein”. Submit your solutions via OLAT by uploading **one** pdf-file with all your solutions **before 10:00 a.m.** Kaiserslautern.

You may submit your solution individually or in a group of at most 2 people. If you opt for a group submission, state the names of both individuals on the first page of the submitted pdf-file.

Exercises with numbers in brackets are taken from the book “An invitation to algebraic geometry” by Smith et. al. (2000).

Exercise 1 [2.3.1]. Show that under the \mathbb{V} and \mathbb{I} operators prime ideals correspond to irreducible algebraic varieties. Conclude that the irreducible components of an affine algebraic variety V correspond to the minimal prime ideals above $\mathbb{I}(V)$.

Exercise 2 . Let $F: V \rightarrow W$ be a morphism of affine algebraic varieties.

- [2.5.1] Show that the pullback $F^\sharp: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ is injective if and only if $F(V)$ is dense in W . We then say that F is a *dominant* morphism.
- [2.5.2] Show that the pullback $F^\sharp: \mathbb{C}[W] \rightarrow \mathbb{C}[V]$ is surjective if and only if F defines an isomorphism between V and some algebraic subvariety of W .

Exercise 3 [2.5.3]. If $F = (F_1, \dots, F_n): \mathbb{A}^n \rightarrow \mathbb{A}^n$ is an isomorphism, then show that the Jacobian determinant

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \quad (0.1)$$

is a nonzero constant polynomial.

(Entertaining fact: It is not known whether the converse is true. This is a famous open problem known as the *Jacobian conjecture*.)

Exercise 4 [2.6.4]. Let $R := \mathbb{C}[x, y]/(x^2)$. Show that $\max\text{Spec}(R)$ is homeomorphic to \mathbb{A}^1 . What is your conclusion?