CHARACTER THEORY OF FINITE GROUPS — **EXERCISE SHEET 1**
RPTU KAISERSLAUTERN-LANDAU — **FB MATHEMATIK**

PROF. DR. ULRICH THIEL
DR. TOBIAS METZLAFF

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SS 2024

Please submit your solution alone or with a (one!) partner in the postbox "Character Theory" on the ground floor, in the office of Dr. Tobias Metzlaff (48-424) or by mail at metzlaff@mathematik.uni-kl.de.

Exercises with numbers in brackets are taken from the book "Introduction to Representation Theory" by Etingof et. al. (2011).

EXERCISE 1 (2.3.15)
Let $V \neq 0$ be a finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation.

EXERCISE 2 (2.3.16)
Let A be an algebra over a field k. Recall that the **center** $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A.

1. Let V be an irreducible finite dimensional representation of A.
   (a) Show that every $z \in Z(A)$ acts on V by multiplication by some scalar $\chi_V(z)$.
   (b) Show that $\chi_V : Z(A) \to k$ is a homomorphism.
2. Let V be an indecomposable finite dimensional representation of A.
   (a) Show that, for all $z \in Z(A)$, the operator $\rho(z)$ by which z acts on V, has only one eigenvalue λ.
   (b) Show that λ is equal to the scalar $\chi_V(z)$ from 1.
3. Does $\rho(z)$ in 2. have to be a scalar operator?

(*Remark*: χ_V is called the **central character** of V.)

EXERCISE 3 (2.5.2)
Let $V \neq 0$ be a representation of A. We say that a vector $v \in V$ is **cyclic** if it generates V, that is, $Av = V$. A representation admitting a cyclic vector is said to be **cyclic**.

1. Show that V is irreducible if and only if all nonzero vectors of V are cyclic.
2. Show that V is cyclic if and only if it is isomorphic to A/I, where I is a left ideal in A.

EXERCISE 4 (2.7.4)
Let $A = k\langle x, y\rangle/\langle yx - xy - 1\rangle$ be the Weyl algebra. If $\mathrm{char}(k) = 0$, what are the finite dimensional representations of A? What are the two-sided ideals in A?
(*Hint*: For the first question, use that $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ for all square matrices B, C. For the second, show that any nonzero two-sided ideal in A contains a nonzero polynomial in x.)