Prof. Dr. Ulrich Thiel
Dr. Tobias Metzlaff

Due date: Thursday, 16.5.2024, 12:00
SS 2024

Please submit your solution alone or with a (one!) partner in the postbox "Character Theory" on the ground floor, in the office of Dr. Tobias Metzlaff (48-424) or by mail at metzlaff@mathematik.uni-kl.de.

Exercises with numbers in brackets are taken from the book "Introduction to Representation Theory" by Pavel Etingof et. al. from 2011 (https://math.mit.edu/ etingof/repb.pdf).

Throughout, k denotes an algebraically closed field.

Exercise 5

Determine the indecomposable representations of the quiver A_2.
(*Hint*: There are three.)

Exercise 6 (2.11.3)

Let V, W, U be k–vector spaces. The terminology "natural" means without choosing a basis.

1. Construct a natural bijection between the bilinear maps from $V \times W$ to U and the linear maps from $V \otimes_k W$ to U.
2. Show that if B is a basis of V and C is a basis of W, then $\{v \otimes w \mid v \in B, w \in C\}$ is a basis of $V \otimes_k W$.
3. For the case that V is finite–dimensional, construct a natural isomorphism from $V^* \otimes W$ to $\mathrm{Hom}(V, W)$.

Exercise 7 (2.15.1)

Recall that a representation of $\mathfrak{sl}(2)$ is a vector space V with operators E, F, H, such that

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H.$$

In this exercise V is a finite dimensional representation over $k = \mathbb{C}$.

(i) Let λ be an eigenvalue of H with maximal real part and generalized eigenspace $\bar{V}_\lambda$. Show that E restricted to $\bar{V}_\lambda$ is 0.

(ii) Let $W \neq 0$ be a representation of $\mathfrak{sl}(2)$ and let $w \in W \setminus \{0\}$ with $E\,w = 0$. For $n > 0$, find a polynomial $P_n(x)$ of degree n, such that $E^n F^n w = P_n(H)\, w$.
(*Hint*: First compute $E F^n w$ and proceed by induction on n.)

(iii) Let $v \in \bar{V}_\lambda$ be a generalized eigenvector of H. Show that there exists $N > 0$ with $F^N v = 0$.

(iv) Show that H is diagonalizable on $\bar{V}_\lambda$.
(*Hint*: Take $N > 0$, such that F^N restricted to $\bar{V}_\lambda$ is 0, and compute $E^N F^N v$ for $v \in \bar{V}_\lambda$.)

(v) Let $N_v > 0$ be the smallest $N > 0$ satisfying (iii). Show that $N_v - 1 = \lambda$.

(vi) Show that for all $N > 0$, there exists an irreducible representation of $\mathfrak{sl}(2)$ of dimension N, which is unique up to isomorphism. Find a suitable basis of this representation to compute the matrices E, F, H.
(*Hint*: Take λ as in (i), $v \in V_\lambda$ an eigenvector of H and consider $v, F\,v, F^2\,v, \ldots$.)