PROF. DR. ULRICH THIEL
DR. TOBIAS METZLAFF

Due date: Thursday, 30.5.2024, 12:00
SS 2024

Please submit your solution alone or with a (one!) partner in the postbox "Character Theory" on the ground floor, in the office of Dr. Tobias Metzlaff (48-424) or by mail at metzlaff@mathematatik.uni-kl.de.

Exercises with numbers in brackets are taken from the book "Introduction to Representation Theory" by Pavel Etingof et. al. from 2011 (https://math.mit.edu/ etingof/repb.pdf).

Throughout, k denotes an algebraically closed field.

EXERCISE 8
Let $A = \mathrm{Mat}_n(k)$. Recall that A^* can be viewed as a left A–module using the duality defined by transposition. Show that A and A^* are isomorphic as A–modules.
(*Hint*: Use the trace map.)

EXERCISE 9
Let A be the algebra of upper triangular $n \times n$ matrices over k. For any i, let V_i be the 1–dimensional representation of A, where a matrix X acts by the scalar X_{ii}. Show that $V_1, \ldots, V_n$ are the irreducible representations of A.

EXERCISE 10 (3.6.1)
Let $W \subseteq V$ be finite dimensional representations of A. Show that $\chi_V = \chi_W + \chi_{V/W}$.

EXERCISE 11 (3.9.5)
Let V be a finite dimensional vector space over $k = \mathbb{C}$ with a symmetric biliner form $(\cdot, \cdot)$. The **Clifford algebra** $\mathrm{Cl}(V)$ is the quotient of the tensor algebra $T(V)$ by the ideal generated by the elements $v \otimes v - (v, v)\,1$, $v \in V$.

Show that if $(\cdot, \cdot)$ is nondegenerate, then $\mathrm{Cl}(V)$ is semisimple and has one irreducible representation of dimension 2^n when $\dim(V) = 2n$ and two such representations when $\dim(V) = 2n + 1$. Then show that there is no other irreducible representation by finding a spanning set of $\mathrm{Cl}(V)$ with $2^{\dim(V)}$ elements.
(*Hint*: For the even case, take a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ of V in which $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}/2$, and construct an irreducible representation of $\mathrm{Cl}(V)$ on $S := \wedge(a_1, \ldots, a_n)$ in which b_i acts by differentiation with respect to a_i. For the odd case, take an additional basis element c with $(c, a_i) = (c, b_i) = 0$, $(c, c) = 1$ that acts on S by ± 1 and admits two nonisomorphic representations S_+, S_-.)