CHARACTER THEORY OF FINITE GROUPS — **EXERCISE SHEET 4**
RPTU KAISERSLAUTERN-LANDAU — **FB MATHEMATIK**

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Due date: Thursday, 13.6.2024, 12:00
SS 2024

Please submit your solution alone or with a (one!) partner in the postbox "Character Theory" on the ground floor, in the office of Dr. Tobias Metzlaff (48-424) or by mail at metzlaff@mathematik.uni-kl.de.

Exercises with numbers in brackets are taken from the book "Introduction to Representation Theory" by Pavel Etingof et. al. from 2011 (https://math.mit.edu/ etingof/repb.pdf).

Throughout, k denotes an algebraically closed field and G a finite group.

EXERCISE 12 (4.1.4)
Let p be a prime number and G be a group of order p^n. Show that every irreducible representation of G over a filed k of characteristic p is trivial.

EXERCISE 13 (4.2.3)
Let $|G| = 0$ in k. Show that the number of isomorphism classes of irreducible representations of G over k is strictly less than the number of conjugacy classes in G.
(*Hint*: For $P = \sum_{g\in G} g \in k[G]$, we have $P^2 = 0$ and the trace of P is 0 in every finite dimensional representation of G over k.)

EXERCISE 14
Read section 4.3 (3) on the irreducible complex representations of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with defining relations

$$i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji, \quad -1 = i^2 = j^2 = k^2.$$

Fill in the missing details. In particular, do the following.

1. Show that the conjugacy classes are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$.
2. Show that $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.
3. Show that the 2–dimensional irreducible representation of Q_8 is given by the Pauli matrices as claimed.

EXERCISE 15 (4.5.2)
Let V_i be the irreducible representations of G over $k = \mathbb{C}$ with characters χ_i and define

$$\psi_i = \frac{\dim(V_i)}{|G|} \sum_{g\in G} \chi_i(g) \cdot g^{-1} \in \mathbb{C}[G].$$

1. Let $v \in V_j$. Show that $\psi_i(v) = \begin{cases} v, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.
   (*Hint*: ψ_i acts on V_j as an intertwining operator, because it commutes with all elements of $\mathbb{C}[G]$. Apply Corollary 2.3.10.)
2. Show that the ψ_i are idempotent, that is, $\psi_i\,\psi_j = \begin{cases} \psi_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.