Character theory of finite groups — **Exercise Sheet 6**
RPTU Kaiserslautern-Landau — **FB Mathematik**

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Please submit your solution alone or with a (one!) partner in the postbox "Character Theory" on the ground floor, in the office of Dr. Tobias Metzlaff (48-424) or by mail at metzlaff@mathematik.uni-kl.de.

Exercises with numbers in brackets are taken from the book "Introduction to Representation Theory" by Pavel Etingof et al from 2011 (https://math.mit.edu/~etingof/reprbook.pdf).

Exercise 20
Show that $\mathbb{A}$ is not Noetherian and that $\overline{\mathbb{Q}}$ is the quotient field of $\mathbb{A}$.

Exercise 21
Let G be a finite group and V be a complex finite-dimensional representation of G. Show that there is a finite field extension K of $\mathbb{Q}$ and a basis of V so that the representing matrices of the group elements with respect to this basis have entries in K.
Remark: We say that "V can be defined over K".
Hint: First show that V can be defined over $\overline{\mathbb{Q}}$.

Exercise 22
Let G be a finite group and V be a complex finite-dimensional representation of G with character χ. Denote by $\mathbb{Q}(\chi)$ the field extension of $\mathbb{Q}$, which is generated by $\{\chi(g),\ g \in G\}$.

1. Show that $\{\chi(g),\ g \in G\} \subseteq \mathbb{A}$.
2. Show that, if V can be defined over a subfield $K \subseteq \mathbb{C}$, then $\mathbb{Q}(\chi) \subseteq K$.
3. Give an example of a representation V, which can not be defined over $\mathbb{Q}(\chi)$.

Exercise 23 (5.2.7)
Let G be a finite group and V be a complex irreducible representation of G with character χ and $\dim(V) \geq 2$. Show that there exists an element $g \in G$ such that $\chi(g) = 0$.