Character Theory of Finite Groups (SS2024)

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Welcome to the course "Character Theory of Finite Groups" in the summer term 2024!

Representation theory is the theory of "linearizing" algebraic structures by "representing their elements as linear operators". Character theory concerns the special (but extremeley important) case of representation theory of finite groups. Hence, the course is actually an introduction to representation theory and as such we will follow a nice book titled "Introduction to representation theory" [1] by Pavel Etingof (et. al.) from 2011 which is available for free at

https://math.mit.edu/~etingof/repb.pdf

The plan is to cover most of the material up to §5.6 (p. 1–104).

The lectures are on **Mondays, 10:00–11:30 in 48-538**, starting on April 22, 2024. There are two exceptions: May 20, 2024 (public holiday); July 8, 2024 (SFB-Begehung Aachen). The target pace is thus about **9 pages per week**.

> I want to emphasize one important aspect. According to the module handbook[a] the effort for this course is 3.0 CP. This amounts to 90h of time, split into 28h presence time (the lectures) and **62h self-study time**. Note that the self-study time is more than twice as much as the presence time! I will require you to work these hours additionally to the lecture! Quoting David P. Roberts: "My job is to make your learning effort as efficient and pleasant as possible, but it is your job to put in the quality time!"
>
> [a] https://modhb.uni-kl.de/mhb/courses/MAT-40-25-K-4/

I will keep a "lecture notebook" below.

Exercises

The course is accompanied by fortnightly exercise classes (1.5 CP) on **Fridays, 11:45–13:15**, starting on May 3, 2024. The tutor (and course assistant) is **Dr. Tobias Metzlaff** (48–424). In the weeks of exercise classes, we will release an exercise sheet on **Monday** after the lecture. Please submit your solutions alone or with a (one!) partner until the following **Thursday by 12:00**. Then on Friday the sheet is dicussed in the exercise class. In order to obtain a certificate ("Übungsschein") for this class, you have to achieve at least **50%** of the points on the exercise sheets in total.

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Hint

For annotating the book (especially with your questions) you can transform the PDF to have larger margins with the tool `pdf-crop-margins`[1] as follows:

```
$pdf-crop-margins -o out.pdf -u -s -p 10 -a4 0 0 -300 0 eti.pdf
```

For annotating PDFs, I recommend the cross-platform open-source software Xournal++[2].

Lecture logbook

Lecture 1 (April 22, 2024)

Pages 1–10 (skipped Section 2.1, which we will integrate later).

I noticed after class that it is better to do Sections 2.4 (Ideals), Section 2.5 (Quotients), and Section 2.6 (Generators and relations) immediately after 2.2 (Algebras) to make the idea of a representation clear right from the start.

I want to summarize this key idea here again. We said a *representation* of a k-algebra A is a k-algebra morphism $\rho\colon A \to \mathrm{End}(V)$, where V is some vector space. So, to any element $a \in A$ we assign a *linear operator* $\rho(a)$. In other words, we *represent* A by linear operators. Concretely, if V is finite-dimensional, then $\mathrm{End}(V) \simeq \mathrm{Mat}_n(k)$, so we represent $a \in A$ by a *matrix*. Since ρ is an algebra morphism, the collection $\{\rho(a) \mid a \in A\}$ of linear operators cannot be arbitrary. For example, they must satisfy $\rho(ab) = \rho(a)\rho(b)$, the latter product being the composition of linear operators. Also, ρ needs to respect all the *relations* in A. To make clear what this means, we use the free algebra.

The free algebra $k\langle x_1, \ldots, x_n\rangle$ satisfies the following *universal property*: if B is any k-algebra and $b_1, \ldots, b_n \in B$ are elements, there is a unique k-algebra morphism $\rho\colon k\langle x_1, \ldots, x_n\rangle \to B$ such that $\rho(x_i) = b_i$. Notice that the (commutative) polynomial ring $k[x_1, \ldots, x_n]$ satisfies the same property for commutative algebras B. Now, let A be any k-algebra. We say that a subset $\boldsymbol{a} = \{a_1, \ldots, a_n\} \subset A$ *generates* A if any element $a \in A$ is a k-linear combination of products of elements of this subset:

$$a = \sum_{\mu \in n^*} \alpha_\mu \mu(\boldsymbol{a}) \ , \tag{0.1}$$

Here, n^* is the set of all finite sequences of elements in our indexing set $\{1, \ldots, n\}$, and if μ is such a sequence, then $\mu(\boldsymbol{a}) \coloneqq \prod_{t \in \mathbb{N}} a_{\mu(t)}$ denotes the product of the corresponding elements in $\boldsymbol{a}$. Moreover, $\alpha_\mu \in k$, and all but finitely many of these scalars are zero so that the expression is finite. The property that $\boldsymbol{a}$ generates A is equivalent to the morphism $\rho\colon k\langle x_1, \ldots, x_n\rangle \to A$, $x_i \mapsto a_i$, being *surjective*. Now, by the homomorphism theorem (which for algebras works the same way as for rings), this induces an isomorphism

$$k\langle x_1, \ldots, x_n\rangle / I \xrightarrow{\simeq} A \ ,$$

where I is the kernel of ρ. Suppose we can write I as the ideal generated by elements $f_1, \ldots, f_m \in A$, so

$$k\langle x_1, \ldots, x_n\rangle / \langle f_1, \ldots, f_m\rangle \xrightarrow{\simeq} A \ .$$

[1] https://pypi.org/project/pdfCropMargins/
[2] https://xournalpp.github.io/

Such an isomorphism is called a *presentation* of A by generators and relations. It is by no means unique! Notice that for simplicity I assumed that we can find *finitely* many generators of A and of the ideal I. This is not true in general but the concept of the free algebra, generators, and relations works the same way with infinitely many generators and I will leave it to you to write this out formally.

Now, if we have such a presentation of A, then by the property of quotients, giving a morphism $\rho\colon A \to B$ into a k-algebra B is the same thing as giving a morphism

$$\rho\colon k\langle x_1, \dots, x_n\rangle \to B$$

such that $I \subset \mathrm{Ker}(\rho)$, i.e. $\rho(f_i) = 0$ for all $i = 1, \dots, m$. By the property of the free algebra, giving a morphism $\rho\colon k\langle x_1, \dots, x_n\rangle \to B$ amounts to giving a collection $b_i := \rho(b_i)$ of elements of $b_i \in B$. Write the f_i in terms of the generators:

$$f_i = \sum_{\mu \in n^*} \alpha_\mu \mu(\boldsymbol{x}) \ ,$$

where $\boldsymbol{x} = \{x_1, \dots, x_n\}$. Then $\rho(f_i) = 0$ translates to

$$0 = \rho(f_i) = \sum_{\mu \in n^*} \alpha_\mu \mu(\boldsymbol{b}) \ ,$$

where $\boldsymbol{b} = \{b_1, \dots, b_n\}$. In other words, giving a morphism $A \to B$ amounts to giving an element of B for each generator of A, and these elements need to satisfy the same relations as the generators. This observation applies in particular to representations, where $B = \mathrm{End}(V)$.

Let us look at the concrete example from Section 2.1. Consider the k-algebra U generated by three elements h, e, f with relations

$$he - eh = 2e \ , \quad hf - fh = -2f \ , \quad ef - fe = h \ .$$

Giving an N-dimensional representation of U amounts to giving three matrices $H, E, F \in \mathrm{Mat}_N(K)$ satisfying the same relations:

$$HE - EH = 2E \ , \quad HF - FH = -2F \ , \quad EF - FE = H \ .$$

Notice that finding such matrices amounts to solving a system of non-linear (in this case quadratic) equations! This is a very hard problem!

Homework

1. Do the introductory Dedekind–Frobenius problem for S_3.
2. Review everything I summarized above.
3. Review modules over a ring. Why is a representation $A \to \mathrm{End}(V)$ the same as a (left) A-module structure on V?
4. Review the group ring.

References

[1] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*. Vol. 59. Student Mathematical Library. With historical interludes by Slava Gerovitch. American Mathematical Society, Providence, RI, 2011, pp. viii+228. DOI: 10.1090/stml/059. URL: https://doi.org/10.1090/stml/059.