Character Theory of Finite Groups (SS2024)

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Welcome to the course "Character Theory of Finite Groups" in the summer term 2024!

What this course is about

Representation theory is the theory of "linearizing" algebraic structures by "representing their elements as linear operators". Character theory concerns the special (but extremely important) case of representation theory of finite groups. Hence, the course is actually an introduction to representation theory.

What we will do

We will follow a nice book titled "Introduction to representation theory" [1] by Pavel Etingof (et. al.) from 2011 which is available for free at

https://math.mit.edu/~etingof/reprbook.pdf

The plan is to cover most of the material up to \$5.6 (pp. 1–104).

Formalities

This module is worth **3.0 CP** (plus additional **1.5 CP** for the exercise classes). More information is in the module handbook https://modhb.uni-kl.de/mhb/courses/MAT-40-25-K-4/.

The lectures are on **Mondays**, **10:00–11:30** in **48-538**, starting on April 22, 2024. There are two exceptions: May 20, 2024 (public holiday); July 8, 2024 (SFB-Begehung Aachen).

Exercise classes

The course is accompanied by fortnightly exercise classes on **Fridays**, **11:45–13:15**, starting on May 3, 2024. The tutor (and course assistant) is **Dr. Tobias Metzlaff** (48–424). In the weeks of exercise classes, we will release an exercise sheet on **Monday** after the lecture. Please submit your solutions alone or with a (one!) partner until the following **Thursday by 12:00**. Then on Friday the sheet is dicussed in the exercise class. In order to obtain a certificate ("Übungsschein") for this class, you have to achieve at least **50%** of the points on the exercise sheets in total.

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Workflow

To make sense of the workflow, I want to emphasize an important aspect. The effort for this course is 3.0 CP. This amounts to 90h of time, split into 28h presence time (the lectures) and **62h self-study time**. Note that the self-study time is more than **4h per week**!

Consequently, each lecture will only be your **first exposure** to a certain amount of material I want you to learn thoroughly by yourself afterwards (the self-study time). Shortly after each lecture, you need to read the section for that lecture in this document and work through everything I tell you before the next lecture. I will have discussed the material in class, but usually not everything and not every detail. For each lecture I will assume you did your homework and that you ask questions if you have difficulties. I will (try to) help you with anything but you need to ask, and for this you need to work.

The best way to learn is to **make things personal**: make your own notes, find (counter)examples, ask questions, spot issues, look up other sources, work (and play) with the stuff!

For annotating PDF files (with your questions, comments, highlights), I recommend the crossplatform open-source software Xournal++¹. I recommend to first transform the file to have larger margins with the tool pdf-crop-margins² as follows:

\$ pdf-crop-margins -o out.pdf -u -s -p 10 -a4 0 0 -300 0 eti.pdf

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¹https://xournalpp.github.io/

²https://pypi.org/project/pdfCropMargins/

1 Lecture 1 (April 22, 2024)

1.1 What you need to work through

Pages 1-10.

1.2 Keywords

The letter from Frobenius to Dedekind. Algebras. The free algebra. The group algebra. Morphisms of algebras. Representations of algebras. A representation of an algebra is the same as a module over the algebra. Regular representation. Representations of the free algebra. Subrepresentations. Irreducible representations.

1.3 Things to think about

- 1. Do the introductory Dedekind–Frobenius problem for S_3 .
- 2. Why is a representation $A \rightarrow End(V)$ the same as a (left) A-module structure on V?
- Maybe it is better to do Sections 2.4 (Ideals), Section 2.5 (Quotients), and Section 2.6 (Generators and relations) immediately after 2.2 (Algebras) to make the idea of a representation clear right from the start. I want to summarize this key idea here again.

We said a *representation* of a *k*-algebra *A* is a *k*-algebra morphism $\rho: A \to \text{End}(V)$, where *V* is some vector space. So, to any element $a \in A$ we assign a *linear operator* $\rho(a)$. In other words, we *represent A* by linear operators. Concretely, if *V* is finite-dimensional, then $\text{End}(V) \simeq \text{Mat}_n(k)$, so we represent $a \in A$ by a *matrix*. Since ρ is an algebra morphism, the collection $\{\rho(a) \mid a \in A\}$ of linear operators cannot be arbitrary. For example, they must satisfy $\rho(ab) = \rho(a)\rho(b)$, the latter product being the composition of linear operators. Also, ρ needs to respect all the *relations* in *A*. To make clear what this means, we use the free algebra.

The free algebra $k\langle x_1, \ldots, x_n \rangle$ satisfies the following *universal property*: if *B* is any *k*-algebra and $b_1, \ldots, b_n \in B$ are elements, there is a unique *k*-algebra morphism $\rho: k\langle x_1, \ldots, x_n \rangle \to B$ such that $\rho(x_i) = b_i$. Notice that the (commutative) polynomial ring $k[x_1, \ldots, x_n]$ satisfies the same property for commutative algebras *B*. Now, let *A* be any *k*-algebra. We say that a subset $\mathbf{a} = \{a_1, \ldots, a_n\} \subset A$ generates *A* if any element $a \in A$ is a *k*-linear combination of products of elements of this subset:

$$a = \sum_{\mu \in n^*} lpha_\mu \mu(oldsymbol{a})$$
 , (1.1)

Here, n^* is the set of all finite sequences of elements in our indexing set $\{1, \ldots, n\}$, and if μ is such a sequence, then $\mu(a) := \prod_{t \in \mathbb{N}} a_{\mu(t)}$ denotes the product of the corresponding elements in a. Moreover, $\alpha_{\mu} \in k$, and all but finitely many of these scalars are zero so that the expression is finite. The property that a generates A is equivalent to the morphism $\rho: k\langle x_1, \ldots, x_n \rangle \to A, x_i \mapsto a_i$, being *surjective*. Now, by the homomorphism theorem (which for algebras works the same way as for rings), this induces an isomorphism

$$k\langle x_1,\ldots,x_n
angle/I\stackrel{\simeq}{\longrightarrow} A$$
 ,

where *I* is the kernel of ρ . Suppose we can write *I* as the ideal generated by elements $f_1, \ldots, f_m \in A$, so

$$k\langle x_1,\ldots,x_n\rangle/\langle f_1,\ldots,f_m\rangle \stackrel{\simeq}{\longrightarrow} A$$
.

Such an isomorphism is called a *presentation* of *A* by generators and relations. It is by no means unique! Notice that for simplicity I assumed that we can find *finitely* many generators of *A* and of the ideal *I*. This is not true in general but the concept of the free algebra, generators, and relations works the same way with infinitely many generators and I will leave it to you to write this out formally.

Now, if we have such a presentation of *A*, then by the property of quotients, giving a morphism $\rho: A \to B$ into a *k*-algebra *B* is the same thing as giving a morphism

$$\rho: k\langle x_1,\ldots,x_n\rangle \to B$$

such that $I \subset \text{Ker}(\rho)$, i.e. $\rho(f_i) = 0$ for all i = 1, ..., m. By the property of the free algebra, giving a morphism $\rho: k\langle x_1, ..., x_n \rangle \to B$ amounts to giving a collection $b_i := \rho(x_i)$ of elements of B. Write the f_i in terms of the generators:

$$f_i = \sum_{\mu \in n^*} lpha_\mu \mu(oldsymbol{x})$$
 ,

where $\mathbf{x} = \{x_1, \dots, x_n\}$. Then $\rho(f_i) = 0$ translates to

$$0=
ho(f_i)=\sum_{\mu\in n^*}lpha_\mu\mu(oldsymbol{b})$$
 ,

where $\mathbf{b} = \{b_1, \dots, b_n\}$. In other words, giving a morphism $A \to B$ amounts to giving an element of B for each generator of A, and these elements need to satisfy the same relations as the generators. This observation applies in particular to representations, where B = End(V).

Let us look at the concrete example from Section 2.1. Consider the k-algebra U generated by three elements h, e, f with relations

$$he - eh = 2e$$
, $hf - fh = -2f$, $ef - fe = h$.

Giving an *N*-dimensional representation of *U* amounts to giving three matrices $H, E, F \in Mat_N(K)$ satisfying the same relations:

$$HE-EH=2E$$
 , $HF-FH=-2F$, $EF-FE=H$.

Notice that finding such matrices amounts to solving a system of non-linear (in this case quadratic) equations! This is a very hard problem!

2 Lecture 2 (April 29, 2024)

2.1 What you need to work through

Pages 11–19.

2.2 Keywords

Morphisms of representations (intertwining operator). Direct sum of representations. Indecomposable representations. The main task of representation theory. Schur's lemma. Irreducible representations of a commutative algebra. Representations of the polynomial ring k[x]. The group algebra. Ideals and quotients of algebras. Algebras defined by generators and relations. The Weyl algebra.

- 1. Review the classification of the irreducible and of indecomposable finite-dimensional representations of the polynomial ring k[x].
- 2. Review the proof of the basis of the Weyl algebra (Proposition 2.7.1). I told you to first look at the action of A on E[t] defined by xf = tf and $yf = \frac{d}{dt}$. Repeat the proof as in the book and notice that the final argument only works if the base field k is of characteristic zero. The proof in the book shows how to make the proof work in any characteristic by considering a more general representation E.

3 Lecture 3 (May 6, 2024)

3.1 What you need to work through

Pages 19-30.

3.2 Keywords

Quivers and their representations. Representations of a quiver is the same as modules over the path algebra. Gabriel's theorem (Theorem 2.1.2). Lie algebras and their representations. Representations of a Lie algebra is the same as modules over the universal enveloping algebra. Representations of $\mathfrak{sl}_2(\mathbb{C})$ (Theorem 2.1.1).

3.3 Things to think about

- 1. Review the fact that representations of a quiver are the same as representations of the path algebra.
- 2. Define the notion of homomorphism, direct sum, subrepresentation, irreducible, indecomposable for quiver representations (see page 21) and notice that they coincide with the notions for representations of the path algebra.
- 3. Study Remark 2.9.4: derivations as "infinitesimal version" of automorphisms.
- 4. Show that the adjoint representation of a Lie algebra is indeed a Lie algebra representation.
- 5. Review the fact that representations of a Lie algebra are the same as representations of its universal enveloping algebra.
- 6. Notice Remark 2.9.3 (Ado's theorem).
- 7. Study Remark 2.9.14 on the connection between Lie groups and Lie algebras.
- 8. Read the historical interlude in Section 2.10.

4 Lecture 4 (May 13, 2024)

4.1 What you need to work through

Pages 31-42.5. Skip the problems and problem sections 2.13-2.16.

4.2 Keywords

Tensor products of vector spaces. Tensors of type (m, n). Tensor algebra, symmetric algebra, exterior algebra, universal enveloping algebra. Semisimple representations of an algebra. If V is a irreducible representation of A, then End(V) is a semisimple representation of A ($\simeq nV$, where $n = \dim(V)$). Description of the subrepresentations of a semisimple representation.

4.3 Things to think about

- 1. We discussed that $V^* \otimes V \simeq \operatorname{End}(V)$ for a finite-dimensional *k*-vector space *V*. More generally, I mentioned (and it is given as an exercise on Sheet 2) that if *V* is finite-dimensional and *W* is any vector space then $V^* \otimes W \simeq \operatorname{Hom}(V, W)$. I want to note that this statement does not hold if *V* is not finite-dimensional, see https://math. stackexchange.com/a/573416.
- 2. Read about the Einstein summation convention for indices on page 32.
- 3. Consider a change of basis on a vector space V, given by an invertible matrix. This induces a change of basis of the dual bases of V*. How does this look like? Now, derive the general transformation rule for (the coefficients of) a tensor. The keywords are *covariance* and *contravariance*. You can read more about this at https://en.wikipedia.org/wiki/Covariance_and_contravariance_of_vectors and https://en.wikipedia.org/wiki/ Tensor. You can also read my notes https://ulthiel.com/math/wp-content/uploads/notes-repository/Covariance-and-Contravariance_annotated.pdf.
- 4. Read Remark 2.11.4 on tensor products $V \otimes_A W$ for a right *A*-module *V* and a left *A*-module *W* (why the left/right?). Notice that $V \otimes_A W$ has no natural *A*-module structure a priori. But it has a natural left *A*-module structure if *V* happens to have a left *A*-module structure as well which is compatible with its right *A*-module structure, i.e. *V* is an *A*-bimodule. This is discussed at length in Problem 2.11.6.
- 5. Do Exercise 2.11.5: if A is an algebra over a field K and L is an extension field of K, then $A^L := L \otimes_K A$ is naturally an algebra over L. If V is an A-module, then $V^L := L \otimes_K V$ is naturally an A^L -module. This process is called *scalar extension*.
- 6. Check that if V is a (left) A-module then End(V) is a (left) A-module as well via (af)(v) = af(v) for $f \in End(V)$, $a \in A$, and $v \in V$. We used this in Example 3.1.2.
- 7. Review the proof that $End(V) \simeq nV$ for an *A*-module *V* of dimension *n* (Example 3.1.2).
- 8. Think deeply about what Proposition 3.1.4 is saying.
- 9. Work through the complete proof of Proposition 3.1.4. The proof could be given some more details, I have discussed this in class.

5 Lecture 5 (May 27, 2024)

5.1 What you need to work through

Pages 42.5-48.

5.2 Keywords

The density theorem. Representations of (products of) matrix algebras. Dual representations. Opposite algebra. Duality on matrix algebras. Filtrations. Existence of filtrations with irreducible quotients. The radical and nilpotent ideals. Structure theorem for finite dimensional algebras: quotient by the radical is a product of matrix algebras. Dimension formula involving the algebra, its radical, and its irreducible representations. Semisimple algebras. Equivalent characterizations of semisimple algebras.

5.3 Things to think about

- Note that part (ii) of Theorem 3.2.2 (density theorem for a direct sum) is not immediately obvious: a direct product of surjective maps is not necessarily surjective, e.g. take the identity id: Z → Z and id × id: Z → Z × Z, n → (n, n).
- 2. Let $\rho: A \to \text{End}(V)$ be an irreducible finite-dimensional representation of an algebra *A*. The density theorem states that if *V* is irreducible, then ρ is surjective. The converse holds as well: if ρ is surjective, then *V* is irreducible. Prove this!
- 3. The (isomorphism classes of) irreducible representations of an algebra A are in one-toone correspondence with those of A/Rad(A). Prove this!
- 4. The proof of Theorem 3.3.1 (representations of matrix algebras) involves some new ideas which are important to think through: the dual representation (which naturally is a *right* module at first) is a *left* module over the *opposite* algebra, and the duality on the matrix algebra given by transposition allows us to make the dual into a *left* module again.
- 5. We defined the radical Rad(A) of a (finite-dimensional) algebra A as the collection of all elements of A which act by zero on all irreducible representations of A. This is more precisely called the *Jacobson radical* of A. One defines it in the same way for *any* ring A. Since irreducible representations of A correspond to maximal left ideals of A (why?), it follows that Rad(A) equals the intersection of all maximal left ideals of A (check this!). One can show that this equals the intersection of maximal *right* ideals of A so that there is no preference of the side. From commutative algebra you may know the nilradical (the collection of all nilpotent elements) and how this relates to the Jacobson radical. For non-commutative rings the connection between the Jacobson radical and nilpotent elements is subtle and led to several notions of radicals, see https://en.wikipedia.org/wiki/Radical_of_a_ring for an introduction.

6 Lecture 6 (Jun 3, 2024)

6.1 What you need to work through

Pages 49–57 (skip the problem section 3.9).

6.2 Keywords

Characters. Character space $(A/[A, A])^*$. Irreducible characters are linearly independent. If the algebra is semisimple, irreducible characters form a basis of the character space. The Jordan–Hölder theorem. The Krull–Schmidt theorem. Representations of tensor products.

1. Let V be a finite-dimensional representation of an algebra A. Let $n_V(S)$ be the multiplicity of a simple representation S as a constituent of V, i.e. the number of times S occurs (up to isomorphism) as a simple subquotient in one (any) composition series of V. Then

$$\chi_V = \sum_S n_V(S)\chi_S$$
 .

Hence, if $p \ge 0$ is the characteristic of the field k, then χ_V determines the multiplicities $n_V(S)$ modulo p (why the mod p?). In particular, if p = 0, then χ_V determines the multiplicities.

- 2. Be aware: there may be non-isomorphic representations with the same character, even if p = 0: consider $A = k[t]/(t^2)$, the module $V = ke_1 \oplus k_2$ with $te_1 = e_2$ and $te_2 = 0$, and the module $W = k^2$ with trivial *t*-action. Then $\chi_V = \chi_W$ but *V* and *W* are not isomorphic.
- 3. Write down a composition series of a semisimple representation $V = \bigoplus_{i=1}^{m} n_i V_i$. What are the constituents, and what are their multiplicities? Show that a composition series is not necessarily unique.
- 4. If p = 0, then semisimple representations are uniquely determined by their character, i.e. if *V* and *W* are semisimple, then $V \simeq W$ if and only if $\chi_V = \chi_W$. Why?
- 5. The proofs of Theorem 3.8.1 (Krull–Schmidt) and Theorem 3.10.3 (Representations of tensor products) may need some further details. Read carefully and try to figure them out.
- 6. In the proof of Theorem 3.10: if $\rho: A \to \operatorname{End}(V)$ and $\sigma: B \to \operatorname{End}(W)$ are surjective, then so is $\rho \otimes \sigma: A \otimes B \to \operatorname{End}(V) \otimes \operatorname{End}(W) \simeq \operatorname{End}(V \otimes W)$. Consider an elementary tensor $f \otimes g \in \operatorname{End}(V) \otimes \operatorname{End}(W)$. Then by surjectivity of ρ and σ there is $a \in A$ and $b \in B$ with $\rho(a) = f$ and $\sigma(b) = g$. Hence, every elementary tensor is in the image of $\rho \otimes \sigma$. But then, by linearity, every tensor is in the image, so $\rho \otimes \sigma$ is surjective.

7 Lecture 7 (Jun 10, 2024)

7.1 What you need to work through

Pages 58-65.

7.2 Keywords

Group representations. Maschke's theorem. Representations of $\mathbb{Z}/p\mathbb{Z}$ in characteristic p. Characters and class functions. Number of irreducible representations equals number of conjugacy classes of G if char(k) does not divide |G|. Complex representations of finite abelian groups, symmetric group S_3 , quaternion group Q_8 , and symmetric group S_4 . Duals and tensor products of representations.

- 1. You really need to understand the examples in Section 4.3. Please work them out completely by yourself.
- Representations of cyclic groups are the underlying principle of the discrete Fourier transform. You can read about this at https://en.wikipedia.org/wiki/Fourier_transform_ on_finite_groups.
- 3. When we proved that irreducible representations of a *commutative* algebra are 1dimensional (Corollary 2.3.12) we used Schur's lemma, and this requires the base field *k* to be *algebraically closed*. Schur's lemma does not hold without this assumption, see Remark 2.3.11. If you want to have some fun you can read https://ulthiel.com/math/wpcontent/uploads/notes-repository/Real-representations-of-cyclic-groups.pdf on the *real* representations of cyclic groups. In this case there are 2-dimensional real irreducible representations. It's fun.
- 4. You can read about Pauli matrices and their role in physics at https://en.wikipedia.org/ wiki/Pauli_matrices.
- 5. Let's explain the relation $\chi_{V\otimes W} = \chi_V \chi_W$ from Section 4.4. In general, if $f: V_1 \to W_1$ and $g: V_2 \to W_2$ are linear maps of vector spaces, we can form the tensor product $f \otimes g$ which is the linear map $V_1 \otimes V_2 \to W_1 \otimes W_2$ defined by $(f \otimes g)(v \otimes w) := f(v) \otimes g(w)$. If you choose bases of V and W, let A be the matrix of f and B be the matrix of g, then the matrix of $f \otimes g$ in the tensor product basis is the so-called Kronecker product $A \otimes B$. I leave it to you to check that

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} .$$

If f and g are endomorphisms, so is $f \otimes g$, and for the trace we get

$$\operatorname{Tr}(f \otimes g) = \operatorname{Tr}(A \otimes B) = a_{11} \operatorname{Tr}(B) + \dots + a_{mn} \operatorname{Tr}(B) = (a_{11} + \dots + a_{nn}) \operatorname{Tr}(B) = \operatorname{Tr}(A) \operatorname{Tr}(B)$$

Now, if $\rho: G \to \text{End}(V)$ and $\sigma: G \to \text{End}(W)$ are representations of G, so is the tensor product $V \otimes W$ via $\rho \otimes \sigma: G \to \text{End}(V \otimes W)$ defined by $(\rho \otimes \sigma)(g) \coloneqq \rho(g) \otimes \sigma(g)$. By the above we have

$$\chi_{V\otimes W}(g) = \mathsf{Tr}\left((
ho\otimes\sigma)(g)
ight) = \mathsf{Tr}(
ho(g)\otimes\sigma(g)) = \mathsf{Tr}(
ho(g))\otimes\mathsf{Tr}(\sigma(g)) = \chi_V(g)\chi_W(g)$$
 .

8 Lecture 8 (Jun 17, 2024)

8.1 What you need to work through

Pages 66-70.

8.2 Keywords

Scalar product on class functions. Scalar product of characters is dimension of the Hom space. Orthogonality relation: irreducible characters form orthonormal basis. "Transposed" orthogonality formula when summing over all characters instead of group elements (involves

the centralizer). Unitary representations. Any finite-dimensional representation of a finite group is unitary with respect to *some* inner product (uniqueness up to scalar for irreducibles). Alternative proof of Maschke's theorem over \mathbb{C} . Matrix elements of an irreducible unitary representation. Orthogonality of matrix elements.

8.3 Things to think about

- 1. Just in case you forgot what $\text{Hom}_G(V, W)$ means in Theorem 4.5.1: these are the morphisms $V \to W$ of representations of G (intertwining operators). Moreover, the vector space isomorphism $\text{Hom}(V, W) \simeq V \otimes W^*$ restricts to $\text{Hom}_G(V, W) \simeq (V \otimes W^*)^G$, the latter denoting the subset of elements that are left invariant by the G-action.
- 2. For the proof of Theorem 4.5.1 note that the element $P \in \mathbb{C}[G]$ acts on a representation X of G, giving an endomorphism $P|_X : X \to X$. Show that the image of $P|_X$ is

$$X^{\mathsf{G}} = \{x \in X \mid gx = x \text{ for all } x \in X\}$$
,

so *P* is the projector on X^G . Show that if *X* is irreducible and not the trivial representation, then $P|_X = 0$. Conclude that for a general representation *X*, the trace $Tr(P|_X)$ is the multiplicity of the trivial representation in *X*. This in turn is equal to the dimension of Hom_{*G*}(\mathbb{C} , *X*).

- 3. Notice that Theorem 4.5.1 means that the *rows* of the character table (see Section 4.8) are orthogonal and Theorem 4.5.4 means that the *columns* are orthogonal. These relations (and more) can sometimes help you to complete a partial character table! See the example of the character table of the alternating group A_4 in Section 4.8.
- 4. Note that the matrix $(t_{ij}(x))_{i,j=1}^n$ formed by the *matrix elements* in Section 4.7 is really just the matrix of the linear operator $\rho_V(x)$ in the chosen basis v_1, \ldots, v_n since $t_{ij}(x) = (\rho_V(x)v_i, v_j)$ is the coefficient of v_i in the basis representation of $\rho_V(x)v_i$.
- 5. For the proof of Proposition 4.7.1 (orthogonality of matrix elements) note a couple of things:
 - (a) The w_i^* form the dual basis of the w_i .
 - (b) Define an inner product on W^* by $(w_i^*, w_i^*) := (w_i, w_j)$.
 - (c) Show that $(xw_i^*, w_i^*) = \overline{(xw_i, x_j)}$. This uses the *G*-invariance of (\cdot, \cdot) .
 - (d) You can define an inner product on $V \otimes W^*$ by $(v \otimes w, v' \otimes w') := (v, v')(w, w')$.
 - (e) $(\chi_{\mathrm{triv}}, \chi_V \otimes \chi_{W^*}) = (\chi_{\mathrm{triv}}, \chi_v \overline{\chi_W}) = (\chi_W, \chi_V).$

9 Lecture 9 (Jun 24, 2024)

9.1 What you need to work through

Pages 71–93 (there are historical interludes in between).

9.2 Keywords

Character tables. Examples S_3 , Q_8 , S_4 , A_5 . Tensor product multiplicities. Frobenius determinant. Frobenius – Schur indicator.

- 1. You have to be able to reproduce the character tables of cyclic groups, S_3 , A_4 , Q_8 , S_4 , and A_5 at any time in your life!
- 2. The motivation for Definition 5.1.1 (complex, real, quaternionic type) is as follows. Given a complex irreducible representation V of a finite group G you can ask: can we realize V over the real numbers, i.e. is there a basis of V such that the matrices of the operators $\rho(g)$ in this basis have real entries for all $g \in G$. A *necessary* condition for this is certainly that χ_V takes only real values. This is equivalent to $\overline{\chi_V} = \chi_V$. Since $\overline{\chi_V} = \chi_{V^*}$, this in turn is equivalent to $V^* \simeq V$. So, if $V \not\simeq V^*$, then certainly V cannot be realized over the real numbers – it is of *complex type*. Let's look further into the case $V \simeq V^*$. The existence of such an isomorphism is equivalent to $\text{Hom}_G(V, V^*) \neq 0$, i.e. there is a nonzero morphism $B: V \to V^*$ of G-modules. But this is equivalent to the existence of a Ginvariant non-zero bilinear form on V. Note that since V is irreducible, $Hom_G(V, V^*) = \mathbb{C}$ if this exists, so up to scalar there will be a unique G-invariant non-zero bilinear form if there is one. One can show (see, e.g., https://math.stackexchange.com/q/4470366) that such a form is either symmetric or skew-symmetric. This is precisely the distinction between real type and quaternionic type. Moreover, one can show (Problem 5.1.2(b)) that V is of real type if and only if V can be realized over the real numbers. Hence, if V being of quaternionic type means that even though χ_V is real-valued, V cannot be realized over the real numbers. Precisely this happens for the 2-dimensional irreducible representation of the quaternionic group!
- 3. Note that in the proof of Theorem 5.1.5 you also have a formula for the Frobenius–Schur indicator: $FS(V) = |G|^{-1} \chi_V(\sum_{g \in G} g^2)$.

10 Lecture 10 (Jul 1, 2024)

10.1 What you need to work through

Pages 94-96.

10.2 Keywords

Algebraic numbers and algebraic integers. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$. Minimal polynomial and algebraic conjugates of an algebraic number.

10.3 Things to think about

- 1. Review Vieta's formulas: https://en.wikipedia.org/wiki/Vieta%27s_formulas
- 2. Review the notion of an algebraically closed field and of the algebraic closure of a field: https://en.wikipedia.org/wiki/Algebraically_closed_field, https://en.wikipedia.org/wiki/ Algebraic_closure
- 3. Check out https://en.wikipedia.org/wiki/Transcendental_number

11 Lecture 11 (Jul 15, 2024)

11.1 What you need to work through

Pages 97-98.

11.2 Keywords

Frobenius divisibility. Solvable groups. Burnside's theorem (statement, started with proof).

11.3 Things to think about

1. What I find astonishing: Burnside's theorem is really about group theory. Yet the proof proceeds via character theory (especially also integrality of some algebraic numbers). On the Wikipedia page https://en.wikipedia.org/wiki/Burnside%27s_theorem you can read the following interesting comment:

The theorem was proved by William Burnside (1904) using the representation theory of finite groups. Several special cases of the theorem had previously been proved by Burnside in 1897, Jordan in 1898, and Frobenius in 1902. John G. Thompson pointed out that a proof avoiding the use of representation theory could be extracted from his work in the 1960s and 1970s on the N-group theorem, and this was done explicitly by Goldschmidt (1970) for groups of odd order, and by Bender (1972) for groups of even order. Matsuyama (1973) simplified the proofs.

There are other theorems like on the so called Frobenius kernel (https://en.wikipedia. org/wiki/Frobenius_group) whose proof is via character theory and no pure group theory proof is known.

However, one could raise the (almost philosophical) question of why the character theory of a group is not considered as group theory and why one seeks alternative proofs. What do you think?

- 2. Recall that a finite group *G* is called *simple* if it has no non-trivial normal subgroups. The Jordan–Hölder theorem states that any finite group *G* admits a *composition series*: a chain $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ of subgroups G_i with G_i normal in G_{i+1} and G_{i+1}/G_i simple. This means that one can think of *G* as being built up from simple groups G_{i+1}/G_i , but note that the precise information about how these simple groups are "stacked on top of each other" to give *G* is only contained in the composition series itself, not just in its simple factor groups. Anyway, this should illustrate that for when we want to understand all finite groups we should begin with the finite *simple* groups. These are indeed classified, see https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups.
- 3. Show that a finite simple group *G* is solvable if and only if it is already abelian. The finite simple *abelian* groups are easy to classify: they are the cyclic groups of prime order $\mathbb{Z}/p\mathbb{Z}$. Hence, when we try to classify finite simple groups, we can focus on the non-abelian ones, and so, by the preceding comment, we can focus on the non-solvable ones. Burnside's theorem now tells us that the order of such a group must be the product of at least *three distinct* prime factors.

12 Lecture 12 (Jul 22, 2024)

12.1 What you need to work through

Pages 99-107.

12.2 Keywords

Burnside's theorem (finished proof). Representations of products of groups (special case of representations of tensor products of algebras, see Theorem 3.10.2). Virtual representations and virtual characters. Lemma on a condition ensuring a virtual character χ_V is an actual character: $(\chi_V, \chi_V) = 1$ and $\chi_V(1) > 0$. Restriction $\text{Res}_H^G V$ of a representation V of G to a subgroup H of G. Induced representation $\text{Ind}_H^G V$. Explicit construction of $\text{Ind}_H^G V$. Dimension of $\text{Ind}_H^G V$. Isomorphism $\text{Ind}_H^G V \simeq \text{Hom}_H(k[G], V)$. Character of $\text{Ind}_H^G V$: Frobenius formula. Only mentioned as statements: Frobenius reciprocity, Clifford theory, Frobenius groups.

12.3 Things to think about

- 1. It is useful to know that $\operatorname{Ind}_{H}^{G}V$ is isomorphic to $k[G] \otimes_{k[H]} V$, see Problem 5.10.2(d) for this.
- 2. Check that Frobenius reciprocity implies for class functions φ and ψ that $(\varphi, \operatorname{Ind}_H^G \psi) = (\psi, \operatorname{Res}_H^G \varphi)$.
- 3. The fundamental concept behind Frobenius reciprocity is an *adjunction* of functors $\operatorname{Ind}_{H}^{G}$: kH-mod $\rightarrow kG$ -mod and $\operatorname{Res}_{H}^{G}$: kG-mod $\rightarrow kH$ -mod. See https://en.wikipedia. org/wiki/Adjoint_functors.
- 4. We have everything ready to compute the character table of the symmetric group S_n . This is done next in the book starting with Section 5.12. I highly recommend to continue reading! It is a beautiful theory connecting representation theory and combinatorics.

References

[1] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*. Vol. 59. Student Mathematical Library. With historical interludes by Slava Gerovitch. American Mathematical Society, Providence, RI, 2011, pp. viii+228. DOI: 10.1090/stml/059. URL: https://doi.org/10.1090/stml/059.