Complex and symplectic reflection groups

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Abstract

Notes for my 4 lectures during the Spring School *Real, Complex, and Symplectic Reflection Groups* at Ruhr-Universität Bochum, March 6-10, 2023.

1 Introduction

Intuitively, all of you know what a *reflection* is:

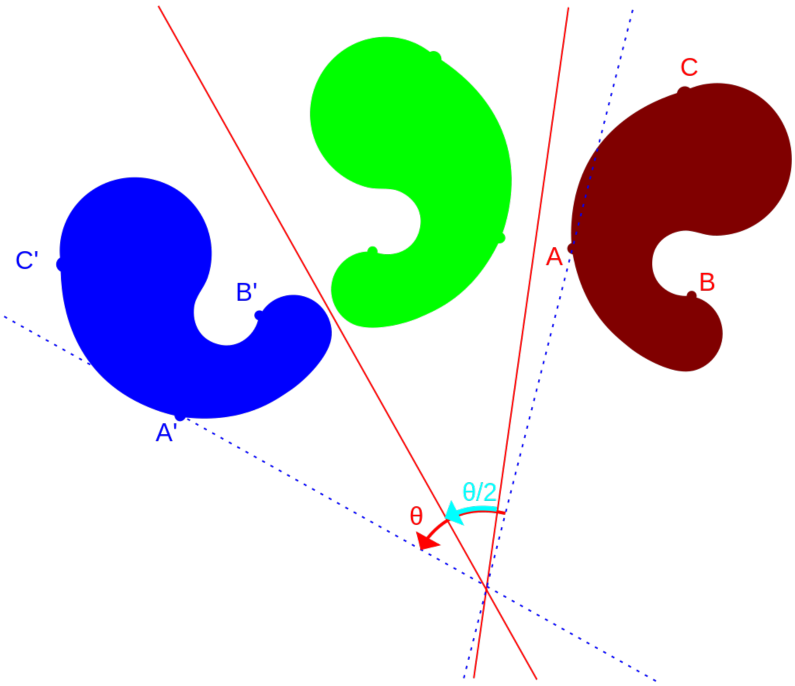

Figure 1: Two reflections in the plane. Source: Wikipedia (CC-BY-2.5)

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Definition 1.1. An *orthogonal reflection* is:

1. an *orthogonal* transformation on $\mathbb{R}^n$ (i.e. a linear map preserving distances and angles),
2. which fixes every point of some *hyperplane* (a subspace of codimension 1),
3. and maps a vector in the orthogonal complement of the hyperplane to its *negative*.

We want to study reflections in greater generality, namely over the *complex numbers* and over the *quaternions* (the latter are also called *symplectic* reflections). The theory is rich and beautiful with countless of connections to a wide range of mathematical fields and many open problems for you to solve.

While the theory of complex reflections is well-established, symplectic reflections are still somewhat exotic. But they gained popularity in the last two decades through applications in algebraic geometry and representation theory.

An excellent reference on complex reflections is the book Lehrer and Taylor 2009. A helpful introduction to symplectic reflections is the thesis Schmitt 2023.

Contents

2 Complex reflection groups

2.1 Preliminary remarks on inner products

Throughout, let V be a *finite-dimensional* vector space over $\mathbb{C}$.

A *inner product* $(\cdot,\cdot)$ on V is a map $(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that:

1. $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ for all $v_1, v_2, w \in V$,
2. $(\lambda v, w) = \lambda (v, w)$ for all $v, w \in V$ and $\lambda \in \mathbb{C}$,
3. $\overline{(v, w)} = (w, v)$ for all $v, w \in V$, where $\bar{\cdot}$ denotes complex conjugation,
4. $(v, v) > 0$ for all $0 \neq v \in V$.

Note that 1-3 implies

$$(v, w_1 + w_2) = (v, w_1) + (v, w_2) \quad \text{and} \quad (v, \mu w) = \overline{\mu}\,(v, w)$$

for all $v, w_1, w_2 \in V$ and $\mu \in \mathbb{C}$. Moreover, 3 implies $(v, v) = \overline{(v, v)}$, so $(v, v) \in \mathbb{R}$ for all $v \in V$ so that 4 makes sense.

We say that $g \in \mathrm{GL}(V)$ leaves $(\cdot,\cdot)$ *invariant*, or that g is a *unitary transformation*, if

$$(gv, gw) = (v, w)$$

for all $v, w \in V$. We denote by $\mathrm{U}(V)$ the subgroup of $\mathrm{GL}(V)$ of unitary transformations. Note that this depends on the choice of $(\cdot,\cdot)$ even though this is not expressed in the notation.

Example 2.1. Let $e_1, \ldots, e_n$ be the *standard basis* of $\mathbb{C}^n$ and let $(\cdot,\cdot)$ be an inner product on $\mathbb{C}^n$. The *Gram matrix* of $(\cdot,\cdot)$ is the matrix J with entries (e_i, e_j). For $v, w \in \mathbb{C}^n$ with coordinates λ_i and μ_i, respectively, we can then write

$$(v, w) = \left(\sum_{i=1}^{n} \lambda_i e_i, \sum_{j=1}^{n} \mu_j e_j \right) = \sum_{i,j=1}^{n} \lambda_i \overline{\mu}_j \,(e_i, e_j) = v^T J \overline{w} \ .$$

The matrix J is *hermitian* ($J = \overline{J}^T$) and *positive definite*. Conversely, any hermitian positive definite matrix defines an inner product by the above formula. In particular, for $J = I$ being the identity we obtain the *standard inner product*

$$\langle v, w \rangle = \sum_{i=1}^{n} \lambda_i \overline{\mu}_i = v^T \overline{w} \ .$$

The linear map $\mathbb{C}^n \to \mathbb{C}^n$ defined by a matrix $M \in \mathrm{GL}_n(\mathbb{C})$ in the standard basis is a unitary transformation with respect to $(\cdot,\cdot)$ if and only if

$$J_{ij} = (e_i, e_j) = (M_i, M_j) = \left(\sum_{k=1}^{n} M_{ik} e_k, \sum_{l=1}^{n} M_{lj} e_j \right) = \sum_{k,l=1}^{n} M_{ki} J_{kl} \overline{M}_{lj} = (M^T J \overline{M})_{ij}$$

i.e.

$$J = M^T J \overline{M} \ . \tag{2.1}$$

In case $J = I$ such a matrix is called a *unitary matrix* (i.e. $I = M^T \overline{M}$) and we denote by $\mathrm{U}_n(\mathbb{C})$ the subgroup of $\mathrm{GL}_n(\mathbb{C})$ of unitary matrices, called the *unitary group*.

We thus have a complete classification of inner products and their unitary transformations on $\mathbb{C}^n$, and thus on any abstract vector space V after choosing a basis.

Even better, the *Gram–Schmidt orthonormalization process* yields for any inner product on $\mathbb{C}^n$ with Gram matrix J an invertible matrix A such that

$$I = A^T J \overline{A} \tag{2.2}$$

i.e. J is transformed to the standard inner product. If the matrix M is unitary with respect to J, i.e. $J = M^T J \overline{M}$, then

$$\begin{aligned} I &= A^T J \overline{A} = A^T \left(M^T J \overline{M} \right) \overline{A} = A^T M^T (A^T)^{-1} A^T J \overline{A} \overline{A}^{-1} \overline{M A} \\ &= (A^{-1} M A)^T I \overline{(A^{-1} M A)} \ , \end{aligned}$$

i.e. $A^{-1} M A$ is a unitary matrix.

Hence, up to change of basis (i.e. conjugation with an invertible matrix) there is just *one* inner product, namely the standard one, and there is just *one* unitary group, namely $\mathrm{U}_n(\mathbb{C})$.

Nonetheless, it is of advantage to work with *abstract* vector spaces and *abstract* inner products because things may look easier with a non-standard inner product (note that orthonormalization will usually introduce square roots). Here is an example.

Lemma 2.2. *Let G be a finite subgroup of $\mathrm{GL}(V)$. Then there is a G-*invariant *inner product on V, i.e., $G \subset \mathrm{U}(V)$ with respect to this inner product.*

Proof. Let $(\cdot, \cdot)$ be any inner product on V (exists by the discussion above) and define a new form by

$$[v, w] = \sum_{g \in G} (gv, gw) \ .$$

Then $[\cdot, \cdot]$ is hermitian and

$$[v, v] = \sum_{g \in G} (gv, gv) > 0$$

is a sum over positive real numbers if $v \neq 0$, so $[\cdot, \cdot]$ is an inner product. Finally, if $h \in G$, then as g runs through G, so does gh and therefore

$$[hv, hw] = \sum_{g \in G} (ghv, ghw) = \sum_{g \in G} (gv, gw) = [v, w] \ ,$$

so $[\cdot, \cdot]$ is G-invariant. □

Example 2.3. Consider the group G in $\mathrm{GL}_2(\mathbb{C})$ generated by

$$s = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \ .$$

These matrices are *not* unitary. You can check that

$$G = \{1, s, t, st, ts, sts = tst\} \ .$$

You can calculate that the Gram matrix J of the G-invariant inner product produced from the standard inner product is

$$J = \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} \ .$$

Now s and t and unitary transformations with respect to this new inner product. The Gram–Schmidt orthonormalization process yields the transformation matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{8}} & \frac{1}{2\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{pmatrix} \ .$$

The matrices s and t above transform to

$$s' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t' = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \ .$$

While these matrices are now unitary, they contain fractions and even square roots.

2.2 Reflections

The *fix space* of $g \in \mathrm{GL}(V)$ is the subspace

$$\{v \in V \mid gv = v\} = \mathrm{Ker}(1 - g) \ .$$

Definition 2.4. A linear transformation $g \in \mathrm{GL}(V)$ is a (complex) *reflection* if it is of *finite order* and its fix space is a *hyperplane*. The hyperplane is called the *reflecting hyperplane*. A reflection in $\mathrm{U}(V)$ is called a *unitary reflection*.

Remark 2.5. The identity is *not* a reflection by definition (its fix space is not a hyperplane but the whole space).

Example 2.6. The linear transformations on $\mathbb{C}^2$ defined by the matrices

$$s = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \ , \quad t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \ , \quad \text{and} \quad sts = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

considered in Example 2.3 are reflections (of order 2). You can compute that

$$H_s = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\rangle \ , \quad H_t = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle \ , \quad H_{sts} = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$$

are the reflecting hyperplanes.

Example 2.7. The fix space of

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a hyperplane but t is of *infinite* order. Such transformations are called *transvections*. The theory of transvections is quite different and we will not consider this here.

Example 2.8. For any root of unity ζ the (1×1)-matrix $(\zeta) \in \mathrm{GL}_1(\mathbb{C})$ is a unitary reflection with (the reflecting hyperplane is the origin). Hence, reflections can be of *arbitrary* (finite) order.

Lemma 2.9. *Let $g \in U(V)$. Then the decomposition*

$$V = \mathrm{Ker}(1-g) \oplus \mathrm{Ker}(1-g)^{\perp} \ , \tag{2.3}$$

*where $(-)^{\perp}$ denotes the orthogonal complement, is g-*stable. *Moreover,*

$$\mathrm{Ker}(1-g)^{\perp} = \mathrm{Im}(1-g) \ . \tag{2.4}$$

Proof. It is clear that $\mathrm{Ker}(1-g)$ is g-stable. If $\alpha \in \mathrm{Ker}(1-g)^{\perp}$, then for $v \in \mathrm{Ker}(1-g)$ we have

$$(g\alpha, v) = (g\alpha, gv) = (\alpha, v) = 0 \ ,$$

so $g\alpha \in \mathrm{Ker}(1-g)^{\perp}$.

Let $u \in \mathrm{Im}(1-g)$, so $u = (1-g)v$ for some $v \in V$. If $w \in \mathrm{Ker}(1-g)$ then

$$\begin{aligned}(u, w) &= (v - gv, w) = (v, w) - (gv, w) = (gv, gw) - (gv, w) \\ &= (gv, gw - w) = (gv, 0) = 0 \ ,\end{aligned}$$

so $u \in \mathrm{Ker}(1-g)^{\perp}$ and therefore $\mathrm{Im}(1-g) \subseteq \mathrm{Ker}(1-g)^{\perp}$. Since

$$\dim(\mathrm{Im}(1-g)) = \dim(V) - \dim(\mathrm{Ker}(1-g)) = \dim \mathrm{Ker}(1-g)^{\perp} \ ,$$

equality follows. $\square$

Since a reflection g is of finite order by definition, the group $\langle g \rangle \subset \mathrm{GL}(V)$ is finite. Hence, by Lemma 2.2 g is *unitary* with respect to *some* inner product $(\cdot, \cdot)$ on V. By Lemma 2.9 we thus have a g-stable decomposition

$$V = \mathrm{Ker}(1-g) \oplus \mathrm{Ker}(1-g)^{\perp} = \mathrm{Ker}(1-g) \oplus \mathrm{Im}(1-g) \ .$$

Since $\mathrm{Ker}(1-g)$ is of dimension $n-1$, the complement $\mathrm{Ker}(1-g)^{\perp} = \mathrm{Im}(1-g)$ is 1-dimensional. A non-zero element α in this space is called a *root* of g. We then have

$$g\alpha = \zeta\alpha$$

for some root of unity ζ of order equal to the order of g and we can thus find a basis of V such that g is the diagonal matrix

$$g = (1, \ldots, 1, \zeta) \ .$$

It follows that a unitary reflection is *uniquely determined* by its reflecting hyperplane (equivalently, by the line orthogonal to it, i.e., a choice of root up to scalar) and by its non-trivial eigenvalue.

The unitary reflection with non-trivial eigenvalue ζ fixing the hyperplane orthogonal to a non-zero vector α is given by the formula

$$r_{\alpha,\zeta}(v) = v - (1-\zeta)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \ . \tag{2.5}$$

Example 2.10. Roots for the reflections

$$s = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \ , \quad t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \ , \quad \text{and} \quad sts = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

considered in Example 2.6 are

$$\alpha_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \alpha_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \quad \alpha_{sts} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ .$$

Note that the roots are not orthogonal to their respective hyperplane with respect to the standard inner product. But they are orthogonal with respect to the inner product computed in Example 2.3.

Remark 2.11. We say that $g \in \mathrm{GL}(V)$ can be *defined* over a subring R of $\mathbb{C}$ if there is a basis of V such that the matrix of g in this basis has entries in R.

Let g be a reflection of order m. It follows from the discussion above that g can be defined over $\mathbb{Z}[\zeta]$ where ζ is a primitive m-th root of unity.

Since the trace of a matrix is invariant under conjugation (change of basis), the trace of g is equal to $n-1+\zeta$, where $d = \dim(V)$. It follows that R must contain ζ, hence $\mathbb{Z}[\zeta]$ is the *minimal subring* of $\mathbb{C}$ over which g can be defined.

In particular, g can be defined over $\mathbb{Z}$ (or, equivalently, $\mathbb{Q}$ or $\mathbb{R}$), if and only if it is of order 2.

Remark 2.12. A *real (orthogonal) reflection* can be defined analogously to Definition 2.4 over the real numbers. Lemma 2.2, Lemma 2.9, and the whole discussion above hold analogously. In particular, a real reflection maps a root to its negative. Hence, this definition coincides with Definition 1.1 from the beginning.

After extending scalars from $\mathbb{R}$ to $\mathbb{C}$, a real (orthogonal) reflection becomes a complex (unitary) reflection of order 2. Conversely, a complex reflection of order 2 is the scalar extension of a real reflection after an appropriate choice of basis.

Some authors (like Bourbaki) say *pseudoreflection* for a complex reflection and mean by *reflection* always a real reflection.

2.3 Reflection groups

Definition 2.13. A (complex) *reflection group*, respectively *unitary reflection group*, is a *finite* subgroup of $\mathrm{GL}(V)$, respectively of $\mathrm{U}(V)$, that is generated by *reflections*.

Remark 2.14. Recall that by Lemma 2.2 any reflection group is a *unitary* reflection group with respect to *some* inner product.

Remark 2.15. We consider the trivial group $\{1\} \subset \mathrm{GL}_n(\mathbb{C})$ as a reflection group (generated by its set of reflections, which is empty).

Remark 2.16. It is not necessarily true that a finite set of reflections (which are of finite order by definition) generate a *finite* group. For example, both

$$s = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are reflections of order 2. But

$$st = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad \text{so} \quad (st)^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} .$$

Remark 2.17. A reflection group is a *matrix group*, i.e., a *subgroup* of some $\mathrm{GL}(V)$. To put the group structure in focus on can also consider an *abstract* finite group G and define a *reflection representation* of G to be a *faithful* (i.e., injective) representation $\rho\colon G \to \mathrm{GL}(V)$ of G such that the image of ρ is generated by reflections.

For $G \subset \mathrm{GL}(V)$ we call the embedding $G \to \mathrm{GL}(V)$ the *natural representation*.

Example 2.18. The linear transformations on $\mathbb{C}^2$ defined by the matrices

$$s = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

considered in Example 2.3 and Example 2.6 generate a *finite* subgroup of $\mathrm{GL}_2(\mathbb{C})$ since $sts = tst$. Since s and t are reflections, this is a reflection group in $\mathrm{GL}_2(\mathbb{C})$. In fact, this is a 2-dimensional reflection representation of the symmetric group S_3. There is one further reflection in this group, namely $sts = tst$.

Example 2.19. Let $\zeta = \zeta_m$ be a primitive m-th root of unity. The linear transformations on $\mathbb{C}^2$ defined by the matrices

$$s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix}$$

are reflections of order 2. We have

$$st = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix} .$$

The order of st is thus equal to m and therefore s and t generate a *finite* subgroup in $\mathrm{GL}_2(\mathbb{C})$. Since s and t are reflections, this is a reflection group. In fact, this is a 2-dimensional reflection representation of the *dihedral group* of order $2m$.

Example 2.20. For $m > 1$ let μ_m be the group of m-th roots of unity, a *cyclic group* of order m. We fix a generator $\zeta = \zeta_m$, i.e., a *primitive* m-th root of unity. Let $\mu_m \to \mathrm{GL}_1(\mathbb{C})$ be the map sending ζ to the (1×1)-matrix (ζ). This is a 1-dimensional unitary reflection representation of μ_m. We denote the image by μ_m^{mat}. Note that any element $\neq 1$ in μ_m^{mat} is a reflection.

Remark 2.21. An abstract group may have *several non-isomorphic* reflection representations. For example, for any k with $\gcd(k, n) = 1$ we have a 1-dimensional reflection representation $\mu_m \to \mathrm{GL}_1(\mathbb{C})$ sending ζ to the matrix (ζ^k), and these representations are pairwise non-isomorphic (their characters are distinct). But note that the image is always the *same* subgroup μ_m^{mat} of $\mathrm{GL}_1(\mathbb{C})$.

Example 2.22. Let $\zeta = \zeta_3$ be a primitive 3rd root of unity. The linear transformations on $\mathbb{C}^2$ defined by the matrices

$$s = \begin{pmatrix} \zeta & 0 \\ \zeta^{-1} & 1 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -\zeta^2 \\ 0 & \zeta \end{pmatrix}$$

are reflections of order 3. Since $sts = tst$, they generate a *finite* subgroup of $\mathrm{GL}_2(\mathbb{C})$. Since s and t are reflections, this is a reflection group. The order of this group is 24 and it is denoted by G_4. This is in fact a 2-dimensional reflection representation of the *binary tetrahedral group* $2T$, which is an extension of the *tetrahedral group* T by a cyclic group of order 2.

2.4 The field of definition

We say that $G \subset \mathrm{GL}(V)$ can be *defined* over a subring R of $\mathbb{C}$ if one can find a basis of V such that the matrix of *every* element of G in this basis has entries in R.

Whereas a single reflection g can always be defined over a ring $\mathbb{Z}[\zeta]$ after Remark 2.11, this is a much more subtle problem for a reflection *group* because one needs to find *one* nice basis for *all* reflections simultaneously.

Continuing Remark 2.11, the minimal possible subfield of $\mathbb{C}$ over which G could *theoretically* be defined is the field $\mathbb{Q}(G)$ generated by the traces of elements of G. We call this the *field of definition* of G.

In Example 2.19 the two reflections s and t are of order two, hence they can each be defined over $\mathbb{Z}$. But they cannot be defined over $\mathbb{Z}$ (or $\mathbb{Q}$) *simultaneously*: we have

$$st = \begin{pmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{pmatrix}$$

which has trace equal to $\zeta + \zeta^{-1} = 2\cos\frac{2\pi}{m}$, which is not a rational number for $m > 2$. So, the field of definition is a *proper* extension of $\mathbb{Q}$ and therefore the group cannot be defined over $\mathbb{Q}$. In fact, $\mathbb{Q}(G) = \mathbb{Q}(\zeta + \zeta^{-1})$ and G can be defined over $\mathbb{Q}(G)$.

In general, a complex representation $\rho\colon G \to \mathrm{GL}(V)$ of a finite group G can always be defined over some *algebraic number field* K. Namely, since the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is algebraically closed, it follows from general Wedderburn theory (see, e.g., Lam 1991, §7) that all complex representations of G can be defined over $\overline{\mathbb{Q}}$. Since, the matrices $\rho(g)$ have just finitely many entries and there are only finitely many such matrices, they generate a *finite* extension K of $\mathbb{Q}$.

It is a classical fact by Brauer 1947 that one can take K to be a *cyclotomic field* $\mathbb{Q}(\zeta)$. Explicitly, $\mathbb{Q}(\zeta_m)$ for m the *exponent* of G will do.

In particular, $\mathbb{Q}(G)$ is a finite extension of $\mathbb{Q}$ contained in some cyclotomic field.

The *Schur index* of G is the minimal number m such that there is a degree-m extension field K of $\mathbb{Q}(G)$ such that G can be defined over K.

If $G \subset \mathrm{GL}(V)$ is a complex reflection group, then G can indeed be defined over $\mathbb{Q}(G)$, i.e., the Schur index is 1. This has been proven by Clark and Ewing 1974.

It is even possible to define G over the *ring of integers* $\mathbb{Z}(G)$ in $\mathbb{Q}(G)$. This has been proven by Nebe 1999.[1]

It is in fact true that *any* complex representation can be defined over $\mathbb{Q}(G)$. This has been proven by Benard 1976. See also Bessis 1997 for a shorter proof.[2]

Remark 2.23. A *real (orthogonal) reflection group* is defined analogously to Definition 2.13 over the real numbers. After extending scalars from $\mathbb{R}$ to $\mathbb{C}$ a real (orthogonal) reflection group becomes a complex (unitary) reflection group all of whose reflections are of order 2. It is a classical fact, see Humphreys 1990, that any real reflection group has the structure of a *finite Coxeter group* and that conversely any finite Coxeter group admits an orthogonal reflection representation, so:

real reflection groups = finite Coxeter groups .

Moreover,

rational reflection groups = finite crystallographic Coxeter groups = Weyl groups.

It is an important theme in the theory of complex reflection groups to study which properties of Coxeter groups can be generalized in some way to complex reflection groups.

Remark 2.24. It is now not surprising anymore that in contrast to a single reflection, a complex reflection group all of whose reflections are of order 2 is *not* necessarily defined over the real numbers. Here is an example.

Let $\zeta = \zeta_8$ be a primitive 8-th root of unity and set $\omega = \zeta^3 + \zeta$. You can check that the linear transformations on $\mathbb{C}^2$ defined by the matrices

$$s = \begin{pmatrix} -1 & 0 \\ -\omega+1 & 1 \end{pmatrix} , \quad t = \begin{pmatrix} 1 & \omega+1 \\ 0 & -1 \end{pmatrix} , \quad u = \begin{pmatrix} \omega-1 & -2 \\ -\omega-1 & -\omega+1 \end{pmatrix}$$

are reflections of order 2. They satisfy the relations $(stu)^4 = (tus)^4 = (ust)^4$ and from this you can conclude that they generate a *finite* subgroup of $\mathrm{GL}_2(\mathbb{C})$. Since the generators are reflections, this is a reflection group. The order of this group is 48

[1] The proof uses the classification of complex reflection groups and case-by-case analysis.

[2] Both proofs use the classification.

and it is denoted by G_{12}. This is in fact a 2-dimensional reflection representation of the *binary octahedral group* $2O$, which is an extension of the *octahedral group* O by a cyclic group of order 2. One can show that there are 12 reflections and all of them are of order 2. But

$$stu = \begin{pmatrix} \omega & -1 \\ -1 & 0 \end{pmatrix}$$

has trace equal to $\omega \notin \mathbb{R}$, hence this group cannot be defined over $\mathbb{R}$.

2.5 The combinatorial reflection groups

2.5.1 The symmetric group

Let $n > 1$ and consider the action of the symmetric group S_n on $\mathbb{C}^n$ by *coordinate permutations*, i.e

$$\sigma e_i = e_{\sigma(i)} \ ,$$

where the e_i are the standard basis vectors.

The matrix M_σ for the action of σ is a *permutation matrix*, i.e., a square matrix which has exactly one non-zero entry in each row and each column, and this entry is equal to 1, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \hat{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix} \ .$$

Conversely, a permutation matrix corresponds to a unique permutation.

We thus have a faithful n-dimensional representation

$$S_n \to \mathrm{GL}_n(\mathbb{C})$$

whose image S_n^{perm} is the *group of permutation matrices* of size n. A permutation matrix is unitary, so this is a subgroup of $\mathrm{U}_n(\mathbb{C})$.

Lemma 2.25. *S_n^{perm} is a unitary reflection group.*

Proof. Let $\sigma = (i, j)$ with $i < j$ be a transposition and let M_{ij} be the corresponding permutation matrix. We have $M_{ij}e_k = e_k$ for all $k \neq i, j$. Moreover, $M_{ij}(e_i+e_j) = e_i+e_j$ and $M_{ij}(e_i - e_j) = e_j - e_i$. Hence, M_{ij} is a reflection with root $e_i - e_j$ and reflecting hyperplane spanned by the e_k for $k \neq i, j$ and by $e_i + e_j$. Since S_n is generated by transpositions, it follows that S_n^{perm} is a reflection group. $\square$

2.5.2 Wreath products

We will now mix the group S_n^{perm} of permutation matrices of size n with the cyclic group μ_m of m-th roots of unity as follows.

The group S_n acts on the product group $\mu_m^n = \mu_m \times \cdots \times \mu_m$ by coordinate permutations

$$\sigma\theta = (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}) \ , \quad \sigma \in S_n, \theta \in \mu_m^n \ .$$

We can now form the *wreath product* $\mu_m \wr S_n$ which is defined as the *semidirect product* $\mu_m^n \rtimes S_n$, i.e., the group with elements of the form $(\theta; \sigma)$ with $\theta \in \mu_m^n$ and $\sigma \in S_n$, and multiplication

$$(\theta; \sigma)(\theta'; \sigma') = (\theta\sigma\theta'; \sigma\sigma') = ((\theta_1\theta'_{\sigma(1)}, \ldots, \theta_n\theta'_{\sigma(n)}); \sigma\sigma') \ .$$

We have an action of $\mu_m \wr S_n$ on $\mathbb{C}^n$ by

$$(\theta; \sigma)e_i = \theta_{\sigma(i)}e_{\sigma(i)} \ .$$

This is really a group action since

$$(\theta; \sigma)\left((\theta'; \sigma')e_i\right) = (\theta; \sigma)\theta'_{\sigma'(i)}e_{\sigma'(i)} = \theta'_{\sigma'(i)}(\theta; \sigma)e_{\sigma'(i)} = \theta'_{\sigma'(i)}\theta_{\sigma\sigma'(i)}e_{\sigma\sigma'(i)}$$

and

$$\left((\theta; \sigma)(\theta'; \sigma')\right)e_i = (\theta\sigma\theta'; \sigma\sigma')e_i = (\theta\sigma\theta')_{\sigma\sigma'(i)}e_{\sigma\sigma'(i)} = \theta_{\sigma\sigma'(i)}\theta'_{\sigma'(i)}e_{\sigma\sigma'(i)} \ .$$

The matrix $M_{(\theta;\sigma)}$ for the action of $(\theta; \sigma)$ is a *generalized permutation matrix* (or *monomial matrix*) with entries in μ_m, i.e., a square matrix which has exactly one non-zero entry in each row and each column, and this entry is contained in μ_m. Conversely, any generalized permutation matrix with entries in μ_m corresponds to a unique pair $(\theta; \sigma)$.

We thus have a faithful n-dimensional representation

$$\mu_m^n \rtimes S_n = \mu_m \wr S_n \to \mathrm{GL}_n(\mathbb{C})$$

whose image $G(m, 1, n)$ is the *group of generalized permutation matrices* of size n with entries in μ_m. A generalized permutation matrix is unitary, so this is a subgroup of $U_n(\mathbb{C})$. The group order is

$$|G(m, 1, n)| = m^n n! \ .$$

Example 2.26. A few familiar cases:

1. $G(1, 1, n) = S_n^{\mathrm{perm}}$.
2. $G(m, 1, 1) = \mu_m^{\mathrm{mat}}$.
3. $G(2, 1, n) \simeq \mu_2 \wr S_n$ is the *group of signed permutations*, also known as the *Weyl group* of type B.

Remark 2.27. Note that $G(m, 1, n)$ is the trivial group for $m = n = 1$. This is why we will assume $mn > 1$ in the following.

Lemma 2.28. *If $mn > 1$, then $G(m, 1, n)$ is a unitary reflection group.*

Proof. The wreath product $\mu_m \wr S_n$ is by definition equal to the semidirect product $\mu_m^n \rtimes S_n$. Every element can thus be uniquely written in the form $\theta\sigma$ for $\theta \in \mu_m^n$ and $\sigma \in S_n$. The symmetric group S_n is generated by transpositions and we already know from Lemma 2.25 that these map to reflections under $\mu_m \wr S_n \to \mathrm{GL}_n(\mathbb{C})$. The element $\theta = (1, 1, \ldots, 1, \zeta)$ maps to a reflection as well. It is now clear that the transpositions together with θ generate $\mu_m^n \rtimes S_n$, hence $G(m, 1, n)$ is a reflection group. □

Example 2.29. The group $G(5, 1, 3)$ is generated by the reflections

$$s = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;, \quad t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;, \quad u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_5 \end{pmatrix} \;.$$

The group is of order 750.

2.5.3 Normal subgroups of wreath products

Let p be a divisor of m (not necessarily prime despite the notation).

Let $\mu_m(p)$ be the subgroup of μ_m generated by ζ_m^p.

This is a subgroup of index p consisting of the $\frac{m}{p}$-th roots of unity.

Consider the map

$$\psi \colon \mu_m^n \to \mu_m \;, \theta = (\theta_1, \ldots, \theta_n) \mapsto \prod_{i=1}^n \theta_i \;.$$

This is a surjective group morphism. We define

$$\mu_m^n(p) = \psi^{-1}(\mu_m(p)) \subset \mu_m^n \;.$$

The group $\mu_m^n(p)$ is normal of index p in μ_m^n.

Hence, $\mu_m^n(p) \rtimes S_n$ is normal of index p in $\mu_m^n \rtimes S_n$.

We define $G(m, p, n)$ as the image of $\mu_m^n(p) \rtimes S_n$ under $\mu_m^n \rtimes S_n \to \mathrm{GL}_n(\mathbb{C})$.

This is the group of generalized permutation matrices with entries in μ_m such that moreover in any matrix the product of the non-zero entries is contained in $\mu_m(p)$.

The group $G(m, p, n)$ is normal in $G(m, 1, n)$ of index p. In particular,

$$|G(m, p, n)| = \frac{m^n n!}{p} \;.$$

Example 2.30. A few familiar cases:

1. $G(2, 2, n) \simeq \mu_2^n(2) \rtimes S_n$ is *group of even-signed permutations*, also known as the *Weyl group* of type D.

2. $G(m,m,2)$ is the dihedral group in the representation as in Example 2.19.

Lemma 2.31. *If $mn>1$ and p is a divisor of m, then $G(m,p,n)$ is a unitary reflection group.*

Proof. By construction we have $G(m,p,n) \simeq \mu_m^n(p) \rtimes S_n$. The transpositions in S_n generate the S_n-part and they map to reflections as discussed before.

We need to find generators for the $\mu_m^n(p)$-part. Start from $(1,\ldots,1)$. Suppose we plug in ζ^k at some position. Then to get this tuple into $\mu_m^n(p)$ we need to plug in $\zeta^{-k}\zeta^{pl}$ for some l at another position.

It follows that $\mu_m^n(p)$ is generated by elements which either have ζ^p in one position (and $=1$ elsewhere) or which contain ζ and ζ^{-1} (and $=1$ elsewhere).

With the help of transpositions it follows that the $\mu_m^n(p)$-part in $\mu_m^n(p) \rtimes S_n$ is generated by $(1,\ldots,1,\zeta^p)$ and by $(1,\ldots,1,\zeta,\zeta^{-1})$. The element $(1,\ldots,1,\zeta^p)$ maps to a reflection. But $(1,\ldots,1,\zeta,\zeta^{-1})$ does not. However, we can replace this generator by $((1,\ldots,1,\zeta,\zeta^{-1});(n-1,n))$, which maps to the reflection

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \zeta \\ & & & \zeta^{-1} & 0 \end{pmatrix}$$

of order 2 (the last (2×2)-block has eigenvalues 1 and -1).

We have shown that $G(m,p,n)$ is generated by reflections. $\square$

Example 2.32. The group $G(8,2,3)$ is generated by the reflections

$$s=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad t=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad u=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_8^2 \end{pmatrix} \quad , \quad v=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_8 \\ 0 & \zeta_8^{-1} & 0 \end{pmatrix} .$$

The group is of order 1536.

Remark 2.33. Whereas $G(m,1,n)$ and $G(m,m,n)$ are generated by n reflections, we need $n+1$ reflections for $G(m,p,n)$ when $p\neq 1,m$.

2.6 The classification

Writing a reflection group $G\subset \mathrm{GL}_n(\mathbb{C})$ in another basis amounts to conjugating G in $\mathrm{GL}_n(\mathbb{C})$. We do not want to distinguish between G and its conjugates, i.e., we are only interested in its *conjugacy class* in $\mathrm{GL}_n(\mathbb{C})$.

We extend this to (abstract) reflection representations.

Definition 2.34. We say that two reflection representations $\rho_1 \colon G_1 \to \mathrm{GL}(V_1)$ and $\rho_2 \colon G_2 \to \mathrm{GL}(V_2)$ are *equivalent* if there is a linear isomorphism $f \colon V_1 \to V_2$ such that

$$f \rho_1(G_1) f^{-1} = \rho_2(G_2) \ .$$

We call the equivalence class of G the *type* of G. Note that this corresponds to a unique conjugacy class of reflection groups in $\mathrm{GL}_n(\mathbb{C})$.

Remark 2.35. Under the conjugation operation $f \circ (-) \circ f^{-1} \colon \ \mathrm{GL}(V_1) \to \mathrm{GL}(V_2)$ every element of $\rho_1(G_1)$ is sent to a unique element of $\rho_2(G_2)$. Since ρ_i is injective, this yields a group isomorphism $\varphi \colon G_1 \to G_2$ with

$$f \rho_1(g_1) f^{-1} = \rho_2(\varphi(g_1))$$

for all $g_1 \in G$. This means the *pullback* $\rho_2 \circ \varphi$ of ρ_2 to G_1 is isomorphic to ρ_1 as representations of G_1.

Note that when we consider a fixed group G and its reflection representations, equivalence means we consider *twisted* isomorphisms of representations of G, twisted by an automorphism of G. This is coarser than usual isomorphisms of representations where the twist is the identity.

Two unitary reflection groups are *unitary equivalent* if f above can be chosen to be an isometry. As in Lemma 2.2 we can turn any reflection group into a unitary one and with the same averaging argument as in the proof a linear isomorphism $V \to V'$ becomes an isometry, so the classification of reflection groups up to equivalence is equivalent to classifying unitary reflection groups up to unitary equivalence.

The classification task was achieved by Shephard and Todd 1954.

FINITE UNITARY REFLECTION GROUPS

G. C. SHEPHARD AND J. A. TODD

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Figure 2: The paper Shephard and Todd 1954.

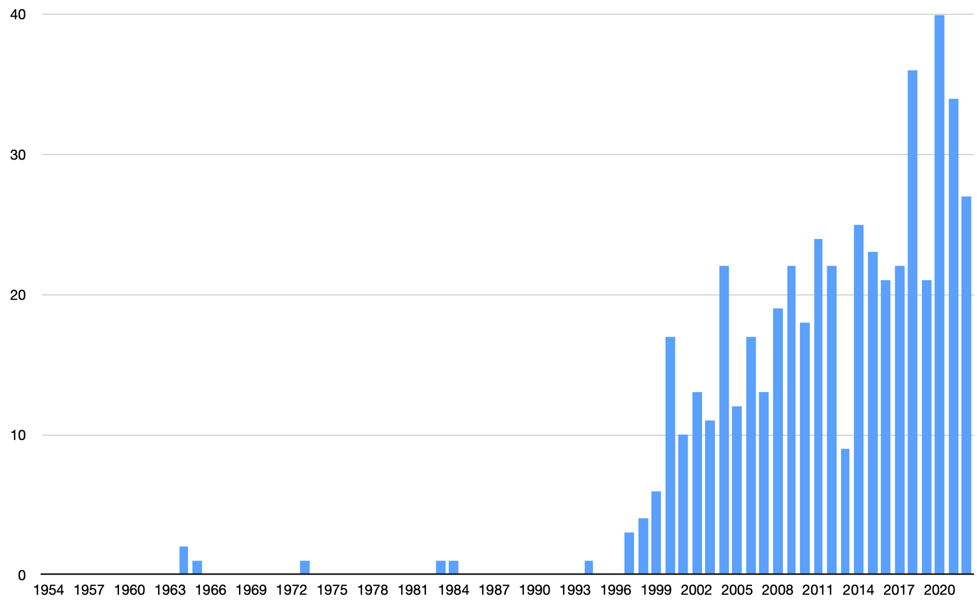

Figure 3: Citations of Shephard and Todd 1954 per year (data from MathSciNet).

2.6.1 Irreducibility

The starting point in the classification is the notion of irreducible groups.

Definition 2.36. A subgroup $G \subset \mathrm{GL}(V)$ is called *irreducible* if the action of G on V is irreducible, i.e., there is no G-stable subspace in V except for V and 0.

Example 2.37. μ_m^{mat} is irreducible for all m (including $m = 1$).

We denote by

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$

the fix space of G.

Proposition 2.38. *Let $G \subset U(V)$ be a unitary reflection group. Then there is an* orthogonal decomposition

$$V = V^G \perp V_1 \perp \ldots \perp V_r$$

of V and a direct product decomposition

$$G = G_1 \times \cdots \times G_r$$

of G such that for each i the factor G_i acts as an irreducible *(unitary) reflection group on V_i and acts as the identity on V_j for all $j \neq i$.*

The space $V_1 \perp \ldots \perp V_r$ is called the *support* of G and its dimension is called the *rank* of G.

The classification of complex reflection groups thus reduces to the classification of the *irreducible* ones.

For the proof of Proposition 2.38 we will need two preliminary facts.

Lemma 2.39. *Let r and s be two unitary reflections with roots α and β, respectively. Then r and s commute if and only if $\mathbb{C}\alpha = \mathbb{C}\beta$ or $(\alpha, \beta) = 0$.*

Proof. We can write $r = r_{\alpha,\zeta}$ and $s = r_{\beta,\eta}$. Then

$$(rs)(v) = v - (1-\zeta)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha - (1-\eta)\frac{(v,\beta)}{(\beta,\beta)}\beta + (1-\zeta)(1-\eta)\frac{(\beta,\alpha)(v,\beta)}{(\alpha,\alpha)(\beta,\beta)}\alpha$$

and similarly

$$(sr)(v) = v - (1-\eta)\frac{(v,\beta)}{(\beta,\beta)}\beta - (1-\zeta)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha + (1-\zeta)(1-\eta)\frac{(\alpha,\beta)(v,\alpha)}{(\beta,\beta)(\alpha,\alpha)}\beta$$

The two expressions are equal if and only if

$$(\beta,\alpha)(v,\beta)\alpha = (\alpha,\beta)(v,\alpha)\beta$$

for all $v \in V$. This holds if and only if α and β are linearly dependent or $(\alpha, \beta) = 0$. $\square$

Lemma 2.40. *Let r be a unitary reflection with root α. A subspace $W \subseteq V$ is r-invariant if and only if $\alpha \in W$ or $\alpha \in W^{\perp}$.*

Proof. If $\alpha \in W^{\perp}$ then $W \subseteq \alpha^{\perp} = \mathrm{Ker}(1-r)$. Hence, r acts as identity on W, so W is r-invariant.

If $\alpha \in W$, then $\mathrm{Im}(1-r) = \mathbb{C}\alpha \subseteq W$. We can write $r = r_{\alpha,\zeta}$. Then

$$r_{\alpha,\zeta}(v) = v - (1-\zeta)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$$

Hence, if $w \in W$, then $r_{\alpha,\zeta}(w) \in w + \mathbb{C}\alpha \subseteq W$, so W is r-stable.

Conversely, assume that W is r-invariant. We can then consider $r|_W \colon W \to W$ and we have $W = \mathrm{Ker}(1 - r|_W) \oplus \mathrm{Im}(1 - r|_W)$. If $\alpha \notin W^{\perp}$, then $W \not\subseteq \mathrm{Ker}(1-r) = \alpha^{\perp}$, so $W \not\subseteq \mathrm{Ker}(1 - r|_W)$. Hence, $0 \neq \mathrm{Im}(1 - r|_W) \subseteq \mathrm{Im}(1-r) = \mathbb{C}\alpha$, so $\alpha \in W$. $\square$

Proof of Proposition 2.38. Let $W \subset V$ be a G-stable subspace. Then $U = W^{\perp}$ is G-stable as well: if $u \in U$ and $w \in W$, then $g^{-1} \in W$ since W is G-stable, hence $(gu, w) = (gu, g(g^{-1}w)) = (u, g^{-1}w) = 0$. Hence, $V = W \perp U$ is a G-stable decomposition. By induction, starting from V^G, we get an orthogonal decomposition $V = V^G \perp V_1 \perp \ldots \perp V_r$ into irreducible G-stable subspaces V_i.

If α is a root of some reflection in G, then $\alpha \in V_i$ for some i by Lemma 2.40. Let G_i be the subgroup of G generated by the reflections with roots in V_i. Then G_i acts as the identity on V_j for $j \neq i$ and G is is generated by the G_i. Since $V_i \perp V_j$ for $j \neq i$, it follows from Lemma 2.39 that G_i and G_j commute. Hence, $G = G_1 \times \ldots \times G_r$. $\square$

Lemma 2.41. *Let $n > 1$. Then for $S_n^{\mathrm{perm}} = G(1,1,n)$ the decomposition from Proposition 2.38 is given by $L \perp L^{\perp}$, where L is the line in $\mathbb{C}^n$ spanned by $e_1 + \ldots + e_n$.*

Proof. Let $W \subseteq \mathbb{C}^n$ be a non-trivial invariant subspace. We claim that either $W = L$ or $W = L^\perp$. Then the action on $L^\perp$ is irreducible, proving the claim. The claim is clear for $n = 2$, so assume $n > 2$. A root for the reflection corresponding to the transposition (i, j) is $e_i - e_j$. It follows from Lemma 2.40 that $e_i - e_j \in W$ or $e_i - e_j \in W^\perp$. Now, if for distinct i, j, k we would have $e_i - e_j \in W$ and $e_j - e_k \in W^\perp$, then $e_j - e_k = 0$, which is a contradiction. So, either $e_i - e_j \in W$ or $e_i - e_j \in W^\perp$ for all $i \neq j$. In the first case we have $L^\perp \subseteq W$ and in the second $W \subseteq L$. Since $\dim(L) = 1$, this forces $L^\perp = W$ or $W = L$. □

In particular S_n^{perm} is *not* irreducible for $n > 1$. A basis for $L^\perp$ is given by the elements

$$b_k = e_k - e_{k+1}$$

for $1 \leq k < n$. The action of the transposition $s_i = (i, i+1)$ on b_k is given by

$$s_i(b_k) = b_k \text{ for } k < i-1 \text{ or } k > i+1 \tag{2.6}$$
$$s_i(b_{i-1}) = e_i - e_{i+1} = e_{i-1} - e_i + e_i - e_{i+1} = b_{i-1} + b_i \tag{2.7}$$
$$s_i(b_i) = e_{i+1} - e_i = -b_i \tag{2.8}$$
$$s_i(b_{i+1}) = e_i - e_{i+2} = e_i - e_{i+1} + e_{i+1} - e_{i+2} = b_i + b_{i+1}\ . \tag{2.9}$$

We denote by $S_n^{\mathrm{ref}} \subset \mathrm{GL}_{n-1}(\mathbb{C})$ the corresponding group. By the discussion above, it is an *irreducible* reflection representation of S_n of rank $n-1$.

Example 2.42. The group S_3^{ref} is generated by

$$s_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\ .$$

We have seen this reflection group several times before.

Remark 2.43. The irreducible reflection representation of S_n just described is isomorphic to the Specht module for the partition $(n-1, 1)$.

Proposition 2.44. *For $m > 1$ the reflection group $G(m, p, n)$ is* irreducible *unless $(m, p, n) = (2, 2, 2)$. The group $G(2, 2, 2)$ is equivalent to $\mu_2^{\mathrm{mat}} \times \mu_2^{\mathrm{mat}}$.*

Proof. The claim is clear for $n = 1$, so we assume $n > 1$. Since $S_n^{\mathrm{perm}} \subset G(m, p, n)$, the only candidates for invariant subspaces are L and $L^\perp$ by Lemma 2.41. It follows from the proof of Proposition 2.38 that L is stable if and only if $L^\perp$ is stable. Recall that $G(m, p, n)$ is generated by the permutation matrices together with

$$s = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \zeta^p \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \zeta \\ & & & \zeta^{-1} & 0 \end{pmatrix},$$

where ζ is a primitive m-th root of unity.

The element $e_1 + \ldots + e_n$ of L is mapped under s to $e_1 + \ldots + \zeta^p e_n$. So, L can be stable only if $\zeta^p = 1$, i.e., $m = p$. Under t it is mapped to $e_1 + \ldots + \zeta e_{n-1} + \zeta^{-1} e_n$. So, L can be stable only if $n = 2$ and $\zeta = \zeta^{-1}$, so $m = 2$. This only leaves the possibility $(m, p, n) = (2, 2, 2)$ and in this case the space L is indeed invariant. The decomposition for $G(2, 2, 2)$ is $L \perp L^\perp$ and the respective restrictions are each equivalent to μ_2^{mat}. □

2.6.2 Imprimivity

Definition 2.45. A subgroup $G \subset \mathrm{GL}(V)$ is said to be *imprimitive* if there is a decomposition $V = V_1 \oplus \ldots \oplus V_r$ into $r > 1$ non-trivial subspaces V_i such that the action of G permutes the V_i. The set $\{V_1, \ldots, V_n\}$ is called a *system of imprimitivity*.

If no such decomposition exists, then G is said to be *primitive*.

The classification of irreducible reflection groups can thus be split into the classification of the imprimitive and of the primitive ones.

Example 2.46. The group μ_m^{mat} is primitive.

Example 2.47. For $n > 1$ the group $G(m, p, n)$ is imprimitive with system of imprimitivity $\{\mathbb{C}e_i \mid 1 \leq i \leq n\}$ since the action of $(\theta; \sigma) \in \mu_m \wr S_n$ maps $\mathbb{C}_i$ to $\mathbb{C}e_{\sigma(i)}$.

The following two facts are not too difficult to prove but we will skip the proof here.

Proposition 2.48. *The group S_n^{ref} is primitive for $n \geq 4$.*

Theorem 2.49. *If $G \subset \mathrm{GL}(V)$ is an irreducible and imprimitive reflection group then G is equivalent to $G(m, p, n)$ for some $n > 1$.*

2.6.3 The primitive groups

The remaining problem—and the main difficulty of the classification—is to determine which other irreducible primitive reflection groups away from μ_m^{mat} and S_n^{ref} there are.

Shephard and Todd 1954 proceed as follows.

Let V be a vector space. The *projective space* $\mathbb{P}(V)$ of V is the set of equivalence classes of all non-zero elements $v \in V$ under the relation $v \sim w$ if $v = \lambda w$ for some $0 \neq \lambda \in \mathbb{C}$.

A *subspace* of $\mathbb{P}(V)$ is the image in $\mathbb{P}(V)$ of a subspace of V.

A *collineation* on $\mathbb{P}(V)$ is a bijective map $\mathbb{P}(V) \to \mathbb{P}(V)$ which preserves inclusions between subspaces in $\mathbb{P}(V)$.

Every linear automorphism on V defines a collineation on $\mathbb{P}(V)$. Such collineations are called *homographies*.[3]

[3] Over the complex numbers there are more collineations than homographies: for example, complex conjugation is a collineation.

A *homology* is a homography defined by a linear automorphism having an eigenspace of dimension $\dim(V) - 1$ and one other eigenvalue.

A reflection group $G \subset \mathrm{GL}(V)$ thus defines a finite group $\mathbb{P}(G)$ of collineations on $\mathbb{P}(V)$ generated by homologies. The group $\mathbb{P}(G)$ is isomorphic to the quotient of G by the subgroup of its scalar matrices.

Collineation groups have been studied extensively before and the primitive ones containing homologies were classified by Blichfeldt 1905 (in dimension 3) and by Mitchell 1914 (in dimension > 3).

In particular, away from $\mathbb{P}(S_n^{\mathrm{ref}})$ (note that the group $\mathbb{P}(\mu_m^{\mathrm{mat}})$ is trivial) there are only *finitely* many and none for $n > 8$.

Shephard and Todd 1954 argue that a collineation group generated by homologies can only come from a finite number of complex reflection groups (up to equivalence) and construct these groups accordingly.

Theorem 2.50 (Shephard and Todd 1954)**.** *Let $G \subset \mathrm{GL}(V)$ be an irreducible complex reflection group. Let $n = \dim(V)$. Then G is equivalent to one of the following:*

1. *the primitive group S_n^{ref} for $n \geq 4$;*
2. *the imprimitive group $G(m,p,n)$ for $m, n > 1$ and $(m,p,n) \neq (2,2,2)$;*
3. *the primitive group μ_m^{mat} for $m \geq 1$ ($n = 1$);*
4. *a primitive group denoted $G_4, \ldots, G_{37}$ by Shephard and Todd ($2 \leq n \leq 8$).*

The proof by Shephard and Todd 1954 is only 5 pages long but draws on the considerable literature on collineation groups generated by homologies.

The groups $G_4, \ldots, G_{37}$ are called the *exceptional* complex reflection groups. They contain several familiar real reflection groups:

$$G_{23} = H_3 \ , \quad G_{28} = F_4 \ , \quad G_{30} = H_4 \ , \quad G_{35} = E_6 \ , \quad G_{36} = E_7 \ , \quad G_{37} = E_8 \ .$$

The orders of the exceptional groups range from 24 for G_4 to 696,729,600 for G_{37}. But note that they are not enumerated by increasing order. We have seen G_4 in Example 2.22 and G_{12} in Remark 2.24.

Remark 2.51. The only overlap in the list in Theorem 2.50 are the following equivalences:

1. $G(4,4,2)$ and $G(2,1,2)$
2. $G(3,3,2)$ and S_3^{ref}
3. $G(2,2,3)$ and S_4^{ref}

Remark 2.52. I find it remarkable that there are so few exceptional groups and that they are all still somewhat manageable with a computer.

Remark 2.53. After the classification, a typical strategy to prove a statement about complex reflection groups is to prove it for the infinite series and then case-by-case for the exceptional groups.

Even though this approach is very powerful, there are two problems:

1. Usually, this does not provide a conceptual understanding of why something is true. (Nonetheless, already knowing that a statement is true may help and motivate to search for a conceptual proof.)

2. One tends to just work with the infinite series (which have a nice combinatorial nature) and stops caring about the few exceptional groups. But then it is not clear whether one ultimately proves a truth about reflection group—the truth is often in the exceptional groups.

Remark 2.54. There is a more systematic approach to the classification by Cohen 1976 based on a generalization of the notion of *root systems* that was used for the classification of real reflection groups. This is also the approach taken by Lehrer and Taylor 2009. Still, all this is far from easy.

Using the classification and computer calculations one can show the following surprising fact which I am not sure has been documented anywhere (but seems to be known!?).

Theorem 2.55 (Thiel 2014)**.** *Two finite irreducible complex reflection groups groups are equivalent if and only if their underlying groups are isomorphic.*

Corollary 2.56. *If $\rho_1 \colon G \to \mathrm{GL}(V)$ and $\rho_2 \colon G \to \mathrm{GL}(V')$ are irreducible reflection representations of a finite group G, then ρ_1 and ρ_2 are equivalent, i.e., ρ_1 is isomorphic to $\rho_2 \circ \varphi$ for some automorphism φ on G.*

Remark 2.57. The statement of the theorem does not hold for reducible groups: the reflection groups $G(6,6,2)$ and $S_3^{\mathrm{ref}} \times \mu_2^{\mathrm{mat}}$ are not equivalent but their underlying groups are isomorphic.

3 Symplectic reflection groups

3.1 The quaternions and quaternionic reflection groups

A *skew-field* (also called *division ring*) is basically the same thing as a field but multiplication may be *non-commutative*.

Most of linear algebra over a field (dimension, matrices, Gaussian elimination, etc.) works also verbatim over a skew-feld.

It is a classical theorem by Frobenius 1878 that the only finite-dimensional skew-fields over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and the *quaternions* $\mathbb{H}$ discovered by Hamilton in 1843.

The quaternions form a 4-dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$, i.e.,

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\} .$$

The multiplication on the elements $\{1, i, j, k\}$ is defined by

$$\begin{gathered} i \cdot 1 = 1 \cdot i = i, \quad j \cdot 1 = 1 \cdot j = j \ , \quad k \cdot 1 = 1 \cdot k = k \ , \\ i^2 = j^2 = k^2 = -1 \ , \\ ij = k, \quad jk = i, \quad ki = j \\ ji = -k \ , \quad kj = -i \ , ik = -j \ . \end{gathered}$$

The multiplication is extended linearly to all of $\mathbb{H}$.

The elements of $\mathbb{R}$ commute with all elements of $\mathbb{H}$. In fact, $\mathbb{R}$ is the center of $\mathbb{H}$.

Since $ij = k$, we can write

$$a + bi + cj + dk = (a + bi) + (c + di)j \ , \tag{3.1}$$

and this shows that $\mathbb{H}$ is a 2-dimensional vector space over $\mathbb{C}$ with basis $\{1, j\}$.

Note that for a complex number $z = a + bi$ we have

$$zj = (a + bi)j = aj + bij = aj - bji = ja - jbi = (a - bi)j = \overline{z}j \ .$$

By a *vector space* V over $\mathbb{H}$ we always mean a right $\mathbb{H}$-module and its *dimension* is always the dimension over $\mathbb{H}$. The group $\mathrm{GL}(V)$ is the group of $\mathbb{H}$-linear automorphisms of V.

Complex conjugation extends to $\mathbb{H}$ by the rule $\overline{j} = -j$, i.e.,

$$\overline{a + bi + cj + dk} = \overline{(a + bi)} + \overline{(c + di)}\overline{j} = (a - bi) - (c - di)j = a - bi - cj - dk \ . \tag{3.2}$$

An *inner product* $(\cdot, \cdot)$ on V is defined as in Section 2.1 and this leads to the subgroup $\mathrm{U}(V)$ of $\mathrm{GL}(V)$ of *unitary* transformations.

As before, an inner product on $\mathbb{H}^n$ is described by its Gram–Matrix J in the standard basis and the linear map $\mathbb{H}^n \to \mathbb{H}^n$ defined by a matrix M is unitary with respect J if and only if $J = M^T J \overline{M}$.

For $J = I$ the identity we obtain the *standard inner product* on $\mathbb{H}^n$ given by

$$\langle v, w \rangle = \sum_{l=1}^{n} \lambda_l \overline{\mu}_l$$

where $\lambda_l, \mu_l \in \mathbb{H}$ are the coordinates of v, w in the standard basis. The corresponding unitary group is denoted by $\mathrm{U}_n(\mathbb{H})$ and consists of matrices M such that $I = M^T \overline{M}$.

With Gram–Schmidt orthonormalization one can show as before that up to change of basis there is just *one* inner product on $\mathbb{H}^n$, namely the standard one.

For any finite subgroup $G \subset \mathrm{GL}(V)$ one can show with the same arguments as in Lemma 2.2 that there is a G-invariant inner product on V.

A (unitary) *quaternionic reflection* is defined exactly as before in Definition 2.4. Similarly as in Equation 3.3 we deduce the formula

$$r_{\alpha,\zeta}(v) = v - (1-\zeta)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \ . \tag{3.3}$$

for a unitary quaternionic reflection. As before, ζ is a root of unity. But this time ζ lives in $\mathbb{H}$ and this is a first complication since there are *many* more roots of unity in $\mathbb{H}$, namely $x\zeta x^{-1}$ for any root of unity $\zeta \in \mathbb{C}$ and $x \in \mathbb{H}$ (recall that the center of $\mathbb{H}$ is $\mathbb{R}$, so there will be elements of $\mathbb{H}$ not commuting with ζ), e.g.,

$$\frac{-1+i+j+k}{2}$$

is a third root of unity.

A *quaternionic reflection group*, respectively *unitary quaternionic reflection group*, is a finite subgroup of $\mathrm{GL}(V)$, respectively of $\mathrm{U}(V)$, generated by quaternionic reflections.

The notions of *irreducible*, *imprimitive*, and (unitary) *equivalent* for (unitary) quaternionic reflection groups are defined as before. Proposition 2.38 about the orthogonal decomposition into irreducible components holds analogously.

The classification of irreducible quaternionic reflection groups up to equivalence was achieved by Cohen 1980.

3.2 Complexification and symplectic reflection groups

3.2.1 Symplectic spaces

For the moment let V be a *complex* vector space again.

A *symplectic form* on V is a bilinear form $\omega\colon V \times V \to \mathbb{C}$ which is:

1. *alternating*, i.e., $\omega(v,v) = 0$ for all $v \in V$;
2. *non-degenerate*, i.e., $\omega(v,w) = 0$ for all $w \in V$ implies $v = 0$.

Note that the alternating condition implies

$$0 = \omega(v+w, v+w) = \omega(v,v) + \omega(v,w) + \omega(w,v) + \omega(w,w) = \omega(v,w) + \omega(w,v) \ ,$$

so

$$\omega(w,v) = -\omega(v,w) \ ,$$

so ω is *skew-symmetric* (this is equivalent to alternating over a field of characteristic $\neq 2$).

We say that $g \in \mathrm{GL}(V)$ leaves ω *invariant*, or that g is a *symplectic transformation*, if

$$\omega(gv, gw) = \omega(v,w)$$

for all $v, w \in V$. We denote by $\mathrm{Sp}(V)$ the subgroup of $\mathrm{GL}(V)$ of symplectic transformations.

Example 3.1. Let ω be a symplectic form on $\mathbb{C}^n$ and let J be its Gram matrix in the standard basis, i.e., $J_{ij} = \omega(e_i, e_j)$. Then

$$\omega(v, w) = v^T J w \ .$$

The matrix J is skew-symmetric ($J^T = -J$) and invertible. Conversely, any such matrix defines a symplectic form. But note that

$$\det(J) = \det(J^T) = \det(-J) = (-1)^n \det(J) \ ,$$

and this can only be true if n is *even*. Hence, a symplectic form can only exist on an even-dimensional space.

So, now suppose we are on $\mathbb{C}^{2n}$. We label the standard basis as $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. The block matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is skew-symmetric and invertible, giving rise to the *standard symplectic form* ω on $\mathbb{C}^{2n}$ which is characterized by the properties

$$\omega(x_i, j_j) = \delta_{ij} \quad \text{and} \quad \omega(x_i, x_j) = 0 \ .$$

The linear map $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$ defined by a matrix $M \in \mathrm{GL}_{2n}(\mathbb{C})$ in the standard basis is a symplectic transformation with respect to a symplectic form ω with Gram matrix J if and only if

$$\Omega = M^T \Omega M \ . \tag{3.4}$$

In case $J = \Omega$ such a matrix is called a *symplectic matrix* and we denote by $\mathrm{Sp}_{2n}(\mathbb{C})$ the subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ of symplectic matrices, called the *symplectic group*.

We thus have a complete classification of symplectic forms and their symplectic transformations on $\mathbb{C}^{2n}$, and thus on any abstract vector space V after choosing a basis.

Even better, a version of the Gram–Schmidt process transforms any symplectic form into the standard one. Hence, up to change of basis there is just *one* symplectic form, namely the standard one, and there is just *one* symplectic group, namely $\mathrm{Sp}_{2n}(\mathbb{C})$.

Example 3.2. The standard symplectic form also has the following interpretation. Let $\mathfrak{h}$ be a complex vector space and let $\mathfrak{h}^*$ be its dual. Then there is a natural symplectic form ω on $\mathfrak{h} \oplus \mathfrak{h}^*$ given by

$$\omega((v, f), (w, g)) = g(v) - f(w) \ .$$

When choosing a basis $\{x_i\}$ of $\mathfrak{h}$ with dual basis $\{y_j\}$, this symplectic form is precisely the standard one.

If $g \in \mathrm{GL}(\mathfrak{h})$, then g induces an automorphism on $\mathfrak{h}^*$. If M is the matrix of g acting on $\mathfrak{h}$ in the basis $\{x_i\}$, then $(M^{-1})^T$ is the matrix of g acting on $\mathfrak{h}^*$ in the dual basis

$\{y_i\}$. Hence, g induces an automorphism $g^{\circledast}$ of $\mathfrak{h} \oplus \mathfrak{h}^*$. It is clear that this leaves the symplectic form ω invariant, i.e., $g^{\circledast} \in \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$.

In particular, a finite subgroup $G \subset \mathrm{GL}(\mathfrak{h})$ defines a finite subgroup $G^{\circledast} \subset \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$.

There is more in this example. When considering $\mathfrak{h}$ as a complex algebraic variety, its cotangent space in every point $v \in \mathfrak{h}$ is precisely $\mathfrak{h}^*$, hence the *cotangent bundle* $T^*\mathfrak{h}$ of $\mathfrak{h}$ is our symplectic space $\mathfrak{h} \oplus \mathfrak{h}^*$. This is the local version of the fact that smooth complex varieties (manifolds) admit a natural symplectic form on their cotangent bundle—a fact that plays a key role in physics.

Definition 3.3. Let (V, ω) be a symplectic space. A *symplectic subspace* is a subspace W of V such that ω restricts to a symplectic form on W.

Remark 3.4. In $\mathbb{C}^n$ with standard symplectic a subspace W is symplectic if and only if it is invariant under multiplication with Ω.

3.2.2 Complexification

We now turn back to quaternions.

A matrix $M \in \mathrm{Mat}_n(\mathbb{H})$ can uniquely be written as $M_1 + M_2 j$ with $M_1, M_2 \in \mathrm{Mat}_n(\mathbb{C})$.

Recall that we consider $\mathbb{H}^n$ as a *right* $\mathbb{C}$-module. We do this to express the *left* action of matrices appropriately.

A vector $v \in \mathbb{H}^n$ can be uniquely written as $v_1 + jv_2$ with $v_1, v_2 \in \mathbb{C}^n$. We set

$$v^\vee = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \; .$$

We call this the *complexification* of quaternionic vectors.

Recall that $jz = \overline{z}j$ for $z \in \mathbb{C}$.

We now obtain:

$$\begin{aligned} Mv &= (M_1 + M_2 j)(v_1 + jv_2) = M_1 v_1 + M_1 j v_2 + M_2 j v_1 + M_2 j^2 v_2 \\ &= M_1 v_1 + j\overline{M}_1 v_2 + j\overline{M}_2 v_1 - M_2 v_2 \\ &= (M_1 v_1 - M_2 v_2) + (\overline{M}_1 v_2 + \overline{M}_2 v_1) \; . \end{aligned}$$

Hence, defining

$$(-)^\vee \colon \mathrm{Mat}_n(\mathbb{H}) \to \mathrm{Mat}_{2n}(\mathbb{C}) \;, \quad M_1 + M_2 j \mapsto \begin{pmatrix} M_1 & -M_2 \\ \overline{M_2} & \overline{M_1} \end{pmatrix} \;, \tag{3.5}$$

we can write

$$(Mv)^\vee = M^\vee v^\vee \; .$$

The map $(-)^\vee$ is called the *complexification* of quaternionic matrices. It is an an injective $\mathbb{C}$-conjugate linear algebra morphism which is compatible with the left action of matrices on vectors. It induces an injective group morphisms

$$\mathrm{GL}_n(\mathbb{H}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \quad \text{and} \quad \mathrm{U}_n(\mathbb{H}) \hookrightarrow \mathrm{U}_{2n}(\mathbb{C}) \ .$$

Now, recall the Gram matrix Ω of the standard symplectic form on $\mathbb{C}^{2n}$. For $M \in \mathrm{Mat}_n(\mathbb{H})$ we compute that

$$\Omega M^\vee = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} M_1 & -M_2 \\ \overline{M_2} & \overline{M_1} \end{pmatrix} = \begin{pmatrix} \overline{M_2} & \overline{M_1} \\ -M_1 & M_2 \end{pmatrix} = \begin{pmatrix} \overline{M_1} & -\overline{M_2} \\ M_2 & M_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \overline{M}^\vee \Omega \ .$$

In particular, if $M \in \mathrm{U}_n(\mathbb{H})$, so that $\overline{M}^{-1} = M^T$, then

$$(M^\vee)^T \Omega M^\vee = \Omega \ , \tag{3.6}$$

i.e., $M^\vee$ is a *symplectic* matrix.

Hence, denoting by

$$\mathrm{USp}_{2n}(\mathbb{C}) = \mathrm{U}_{2n}(\mathbb{C}) \cap \mathrm{Sp}_{2n}(\mathbb{C})$$

the *unitary symplectic group*, we conclude that

$$\mathrm{U}_n(\mathbb{H})^\vee = \mathrm{USp}_{2n}(\mathbb{C}) \ .$$

If W is a subspace of V, then $W^\vee$ is a subspace of $V^\vee$. Since W is stable under multiplication with j and $(jI)^\vee = -\Omega$, it follows that $W^\vee$ is stable under multiplication with Ω, so $W^\vee$ is a *symplectic* subspace of $V^\vee$. Conversely, any symplectic subspace of $V^\vee$ is the complexification of a subspace of V.

The notions of *irreducible*, *imprimitive*, *equivalent*, and *orthogonal decomposition* for a subgroup of $\mathrm{U}_n(\mathbb{H})$ translate under complexification to respective properties on the complex symplectic side. We extend them to general finite subgroups of automorphisms of a complex symplectic space and then usually use the prefix “symplectically”. We use the prefix “complex” when forgetting about the symplectic structure.

3.2.3 Symplectic reflection groups

Let $g \in \mathrm{U}_n(\mathbb{H})$ be a unitary quaternionic reflection.

The discussion above shows that $g^\vee$ is a unitary symplectic automorphism on $\mathbb{C}^{2n}$.

The fixed space of g in $\mathbb{H}^n$ is of codimension 1 over $\mathbb{H}$, hence the fixed space of $s^\vee$ in $\mathbb{C}^{2n}$ is of codimension *2* over $\mathbb{C}$.

This brings us to the following definition:

Definition 3.5. Let V be a complex vector space equipped with a symplectic form ω. A *symplectic reflection* is a symplectic automorphism $g \in \mathrm{Sp}(V)$ whose fixed space in V is of codimension 2.

A *symplectic reflection group* is a finite subgroup $G \subset \mathrm{Sp}(V)$ generated by symplectic reflections.

If $G \subset \mathrm{U}_n(\mathbb{H})$ is a unitary quaternionic reflection group, then $G^\vee \subset \mathrm{USp}_{2n}(\mathbb{C})$ is a unitary symplectic reflection group.

Conversely, any unitary symplectic reflection group in $\mathrm{USp}_{2n}(\mathbb{C})$ is the complexification of a quaternionic reflection group in $\mathrm{U}_n(\mathbb{H})$.

Example 3.6. If $G \subset \mathrm{GL}(\mathfrak{h})$ is a complex reflection group then $G^\circledast \in \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is a symplectic reflection group. The symplectic reflections in $G^\circledast$ are of the form $s^\circledast$ for $s \in \mathrm{GL}(\mathfrak{h})$ a reflection.

There is one subtlety: given a symplectic reflection group $G \subset \mathrm{Sp}(V)$ we can choose a basis such that G becomes a subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$ and we can choose a basis such that G becomes a subgroup of $\mathrm{U}_{2n}(\mathbb{C})$. But to connect this to the unitary quaternionic side, we need to have both *simultaneously*. This is indeed possible thanks to the following proposition.

Proposition 3.7 (Cohen 1980). *Let G be a finite irreducible subgroup of $\mathrm{U}_{2n}(\mathbb{C})$. Then the following are equivalent:*

1. *There is a G-invariant symplectic form on $\mathbb{C}^{2n}$.*
2. *G is conjugate to a subgroup of $\mathrm{USp}_{2n}(\mathbb{C})$.*

Hence, quaternionic reflection groups and symplectic reflection groups are the same thing and their classifications are equivalent.

It is more convenient to work with symplectic groups over the complex numbers because then we can do invariant theory etc. as usual whereas this is not established over a non-commutative base ring.

3.3 The classification

The classification of symplectic reflection groups splits into the branches as depicted in Figure 4. A detailed description of the groups is given in Cohen 1980 (see also Schmitt 2023 for a summary).

3.4 An Application

Let V be a finite-dimensional complex vector space and let $G \subset \mathrm{GL}(V)$ be a finite group.

We can consider the vector space V as a complex algebraic variety. The coordinate ring of V, i.e., the ring of polynomial functions on V, is

$$\mathbb{C}[V] = \mathrm{Sym}(V^*) \ ,$$

where $\mathrm{Sym}(V^*)$ is the *symmetric algebra* of the dual space V^*, i.e., the quotient of the *tensor algebra* of V^* by the commutator relations. When choosing a basis $y_1, \ldots, y_n$ of V and denoting by $x_1, \ldots, x_n$ its dual basis of V^*, then

$$\mathbb{C}[V] \simeq \mathbb{C}[x_1, \ldots, x_n]$$

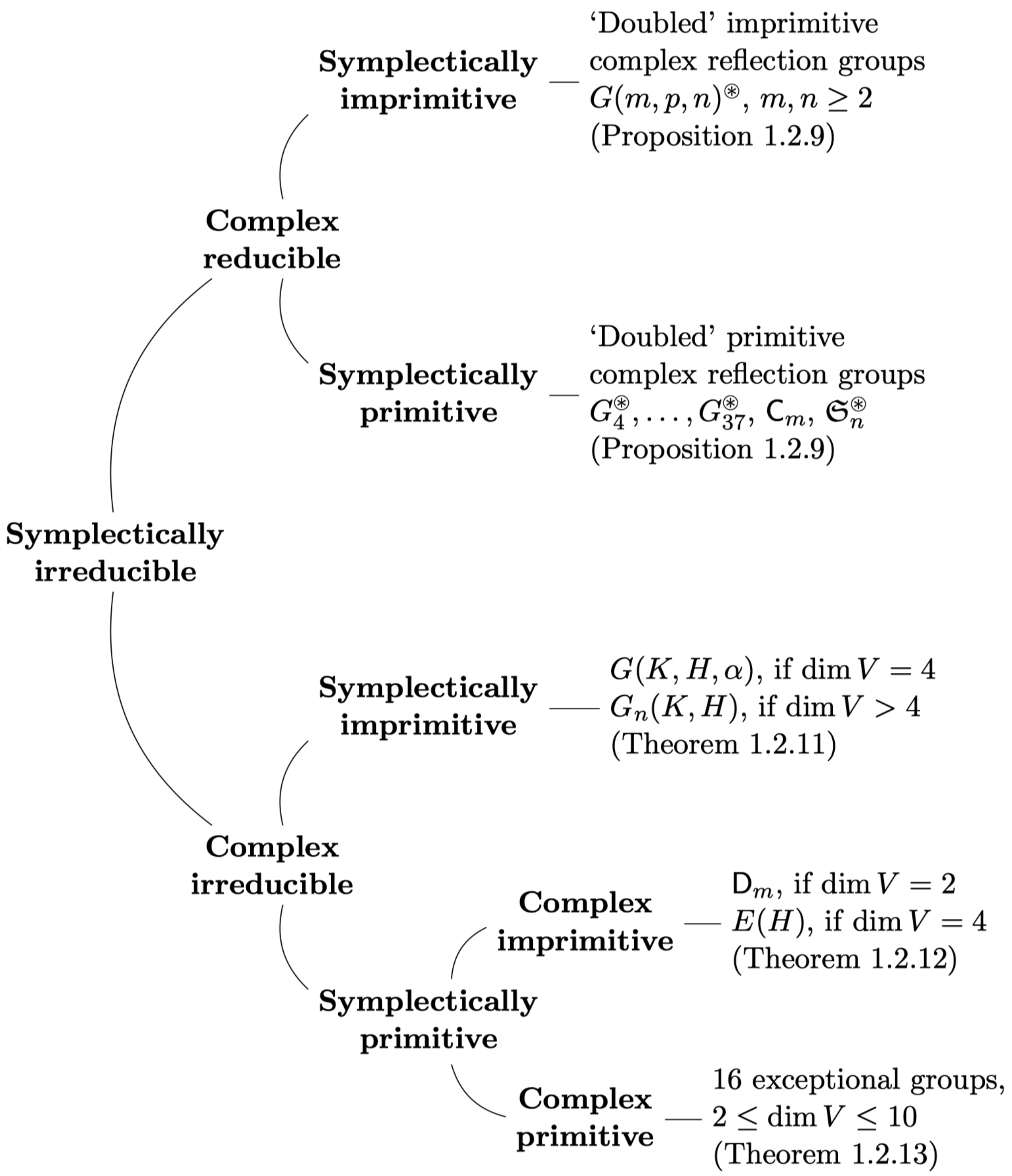

Figure 4: The classification of symplectic reflection groups. Source: Schmitt 2023.

as $\mathbb{C}$-algebras.

The group G acts on V and this induces an action on V^* via

$$(gf)(v) = f(g^{-1}v)$$

for $g \in G$, $f \in V^*$, and $v \in V$.

If M_g is the matrix of the action of g on V in the basis $y_1, \ldots, y_n$ of V, then $(M_g^T)^{-1}$ is the matrix of the action of g on V^* in the dual basis $x_1, \ldots, x_n$.

Extending the action of G on V^* multiplicatively we obtain an action of G on $\mathbb{C}[V]$, i.e.,

$$g(f_1 \ldots f_r) = (gf_1) \cdots (gf_r)$$

for $f_i \in V^*$. This action is by (graded) $\mathbb{C}$-algebra automorphisms.

The *invariant ring* of G is

$$\mathbb{C}[V]^G = \{f \in \mathbb{C}[V] \mid gf = f \text{ for all } g \in G\} \ .$$

It is clear that this is indeed a subring of $\mathbb{C}[V]$.

The following are elementary facts from commutative algebra by Hilbert and Noether.

Proposition 3.8. *The ring extension $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ is* integral *and $\mathbb{C}[V]^G$ is a* finitely generated *$\mathbb{C}$-algebra.*

Since $\mathbb{C}[V]^G$ is a finitely generated $\mathbb{C}$-algebra and integral (as a subalgebra of $\mathbb{C}[V]$), we can associate an irreducible complex variety X to this ring. Since $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ is integral, we get an induced closed surjective morphism $V \to X$. One can show that X is as a set equal to the orbit space V/G and that $V \to V/G$ is the quotient map.

Theorem 3.9 (Shephard–Todd, Serre)**.** *For a finite subgroup $G \subset \mathrm{GL}(V)$ the following are equivalent:*

1. *G is a (complex) reflection group.*
2. *The invariant ring $\mathbb{C}[V]^G$ is a polynomial ring (i.e., has an algebraically independent generating set).*
3. *The variety V/G is* smooth*, i.e., the localization of $\mathbb{C}[V]^G$ in any maximal ideal is a* regular *local ring.*

Now, suppose that V is symplectic and that $G \subset \mathrm{Sp}(V)$.

Since $\mathrm{Sp}(V) \subseteq \mathrm{SL}(V)$, there are *no* reflections in G.

In particular, if $G \neq 1$, then G is not a reflection group and therefore V/G has singularities.

The singularities are of an interesting type: they are *symplectic singularities*.

A (projective) *symplectic resolution* of V/G is a (projective) resolution of singularities $\pi\colon X \to V/G$ such X is a smooth symplectic variety and π is an isomorphism of symplectic varieties over the smooth part of V/G.

Theorem 3.10 (Verbitsky). *If V/G admits a symplectic resolution, then G is generated by* symplectic reflections.

Example 3.11. $(T^*\mathbb{C}^n)/\{\pm 1\}$ does not have a symplectic resolution if $n > 1$.

For which symplectic reflection groups $G \subset \mathrm{GL}(V)$ does V/G admit a symplectic resolution?

Work by several authors over the last two decades using the classification of symplectic reflection groups showed that symplectic resolutions exists only rarely: for example, among the $G^{\circledast}$ for irreducible complex reflection group G only S_n^{perm}, $G(m,1,n)$, μ_m^{mat} and G_4 admit one.

Latest achievement by Bellamy, Schmitt, and Thiel 2022; Bellamy, Schmitt, and Thiel 2023: reduction to finitely many open cases (45 cases, all in dimension 4).

The invariant theory of a symplectic reflection group $G \subset \mathrm{GL}(V)$ is very complicated and not understood. This already starts with the groups $G^{\circledast}$ for a complex reflection group G.

An important classical fact in the theory of complex reflection groups is the *Steinberg fixed point theorem.*

Theorem 3.12 (Steinberg 1964). *Let $G \subset \mathrm{GL}(V)$ be a complex reflection group. Then for any $v \in V$ the stabilizer subgroup G_v of v is again a complex reflection group (generated by the reflections which fix v).*

The proof in Steinberg 1964 uses the classification.

Theorem 3.13 (Bellamy, Schmitt, and Thiel 2023). *Let $G \subset \mathrm{GL}(V)$ be a symplectic reflection group. Then for any $v \in V$ the stabilizer subgroup G_v of v is again a symplectic reflection group (generated by the symplectic reflections which fix v).*

Corollary 3.14. *If $G \subset \mathrm{GL}(V)$ is a symplectic reflection group then the singular locus of V/G is of pure codimension two.*

> Note that the class of complex reflection groups contains the class of finite Coxeter groups (the real reflection groups), and thus the class of Weyl groups (the rational reflection groups). The idea behind the "spetses" program initiated by Broué, Malle, and Michel Broué, Malle, and Michel 1999 is that it seems there are "fake" algebraic groups associated not just to Weyl groups but to complex reflection groups in general. Given that the class of symplectic reflection groups contains the class of complex reflection groups I find the following question intriguing: do some parts of the "spetses" program make it to the larger class of symplectic reflection groups? Surely not everything, maybe nothing—but maybe something!?
> — Thiel 2021

References

Bellamy, Gwyn, Johannes Schmitt, and Ulrich Thiel (2022). “Towards the classification of symplectic linear quotient singularities admitting a symplectic resolution.” In: *Math. Z.* 300.1, pp. 661–681. ISSN: 0025-5874. DOI: 10.1007/s00209-021-02793-9. URL: https://doi.org/10.1007/s00209-021-02793-9.

– (2023). “On parabolic subgroups of symplectic reflection groups.” In: *Glasg. Math. J.* To appear. DOI: https://www.doi.org/10.1017/S0017089522000416.

Benard, Mark (1976). “Schur indices and splitting fields of the unitary reflection groups.” In: *J. Algebra* 38.2, pp. 318–342. ISSN: 0021-8693. DOI: 10.1016/0021-8693(76)90223-4. URL: https://doi.org/10.1016/0021-8693(76)90223-4.

Bessis, David (1997). “Sur le corps de définition d’un groupe de réflexions complexe.” In: *Comm. Algebra* 25.8, pp. 2703–2716. ISSN: 0092-7872. DOI: 10.1080/00927879708826016. URL: https://doi.org/10.1080/00927879708826016.

Blichfeldt, H. F. (1905). “The finite, discontinuous primitive groups of collineations in four variables.” In: *Math. Ann.* 60.2, pp. 204–231. ISSN: 0025-5831. DOI: 10.1007/BF01677268. URL: https://doi.org/10.1007/BF01677268.

Brauer, Richard (1947). “Applications of induced characters.” In: *Amer. J. Math.* 69, pp. 709–716. ISSN: 0002-9327. DOI: 10.2307/2371795. URL: https://doi.org/10.2307/2371795.

Broué, M., G. Malle, and J. Michel (1999). “Towards spetses. I.” In: vol. 4. 2-3. Dedicated to the memory of Claude Chevalley, pp. 157–218. DOI: 10.1007/BF01237357. URL: https://doi.org/10.1007/BF01237357.

Clark, Allan and John Ewing (1974). “The realization of polynomial algebras as cohomology rings.” In: *Pacific J. Math.* 50, pp. 425–434. ISSN: 0030-8730. URL: http://projecteuclid.org/euclid.pjm/1102913229.

Cohen, Arjeh M. (1976). “Finite complex reflection groups.” In: *Ann. Sci. École Norm. Sup. (4)* 9.3, pp. 379–436. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1976_4_9_3_379_0.

– (1980). “Finite quaternionic reflection groups.” In: *J. Algebra* 64.2, pp. 293–324. ISSN: 0021-8693. DOI: 10.1016/0021-8693(80)90148-9. URL: https://doi.org/10.1016/0021-8693(80)90148-9.

Frobenius, Herrn (1878). “Ueber lineare Substitutionen und bilineare Formen.” In: *J. Reine Angew. Math.* 84, pp. 1–63. ISSN: 0075-4102. DOI: 10.1515/crelle-1878-18788403. URL: https://doi.org/10.1515/crelle-1878-18788403.

Humphreys, James E. (1990). *Reflection groups and Coxeter groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xii+204. ISBN: 0-521-37510-X. DOI: 10.1017/CBO9780511623646. URL: https://doi.org/10.1017/CBO9780511623646.

Lam, T. Y. (1991). *A first course in noncommutative rings*. Vol. 131. Graduate Texts in Mathematics. Springer-Verlag, New York, pp. xvi+397. ISBN: 0-387-97523-3. DOI: 10.1007/978-1-4684-0406-7. URL: https://doi.org/10.1007/978-1-4684-0406-7.

Lehrer, Gustav I. and Donald E. Taylor (2009). *Unitary reflection groups*. Vol. 20. Australian Mathematical Society Lecture Series. Cambridge University Press, Cambridge, pp. viii+294. ISBN: 978-0-521-74989-3.

Mitchell, Howard H. (1914). “Determination of All Primitive Collineation Groups in More than Four Variables which Contain Homologies.” In: *Amer. J. Math.* 36.1, pp. 1–

12. ISSN: 0002-9327. DOI: 10.2307/2370513. URL: https://doi.org/10.2307/2370513.

Nebe, Gabriele (1999). "The root lattices of the complex reflection groups." In: *J. Group Theory* 2.1, pp. 15–38. ISSN: 1433-5883. DOI: 10.1515/jgth.1999.001. URL: https://doi.org/10.1515/jgth.1999.001.

Schmitt, Johannes (2023). "Birational geometry and representation theory of some symplectic linear quotient singularities." PhD thesis. RPTU Kaiserslautern-Landau.

Shephard, G. C. and J. A. Todd (1954). "Finite unitary reflection groups." In: *Canad. J. Math.* 6, pp. 274–304. ISSN: 0008-414X. DOI: 10.4153/cjm-1954-028-3. URL: https://doi.org/10.4153/cjm-1954-028-3.

Steinberg, Robert (1964). "Differential equations invariant under finite reflection groups." In: *Trans. Amer. Math. Soc.* 112, pp. 392–400. ISSN: 0002-9947. DOI: 10.2307/1994152. URL: https://doi.org/10.2307/1994152.

Thiel, Ulrich (2014). "On restricted rational Cherednik algebras." PhD thesis. University of Kaiserslautern.

– (2021). "Geometry and representation theory associated to symplectic reflection groups." In: *Computational Group Theory*. Vol. Oberwolfach Reports 38, pp. 16–19.