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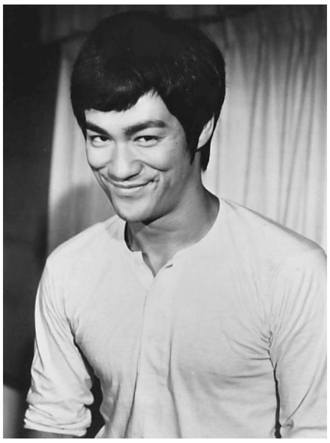
What are "Lie Algebras" about?

First, a clarification:

(pictures from Wikipedia, public domain)



Bruce Lee (1940-1973)

Sophus Lie (1842-1899)

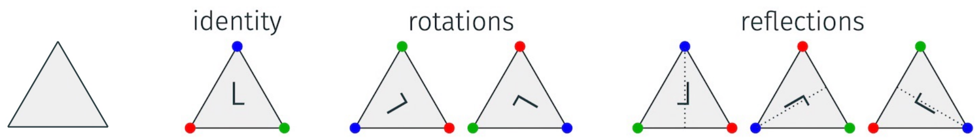
Notice the difference in spelling!

I guess you remember what a <u>group</u> is. I'm sure you also know where this concept comes from:

Symmetry

E.g. every polynomial has a Galois group permuting the roots. A geometric object has a group of symmetries leaving the object unchanged, e.g. for a triangle in the plane

(picture from own TikZ)



6 symmetries (dihedral group of order 6 $\simeq$ symmetric group S_3)

Knowing about symmetries is very important as it helps to simplify things.
We mathematicians can study symmetry without necessarily having a concrete object/problem in mind. Can do this by studying an (abstract) group G. This brings us to the following concept.

A (linear) <u>representation</u> of a group G is a group morphism $\rho: G \longrightarrow GL(V)$ for some vector space V. Note that since ρ is a group morphism, every relation that holds in G, also holds between the linear operators $\rho(g)$, $g \in G$, on V, i.e. the symmetry encoded abstractly in G is now more concretely also present in these linear operators. So, basically, a representation is a linear object obeying the symmetries encoded in G.

You know what a subspace $U \leq V$ is, right? So, what's a <u>subrepresentation</u> of ρ?
It's a subspace $U \leq V$ which is stable under all the operators $\rho(g)$, $g \in G$.

Here's an example: consider the representation

$$\begin{aligned} \rho : \underset{\text{symmetric group}}{S_3} &\longrightarrow GL_3(\mathbb{R}) = GL(\mathbb{R}^3) \\ \sigma &\longmapsto (\underset{\text{i-th standard basis vector}}{e_i} \mapsto e_{\sigma(i)}) \end{aligned}$$

Let $U \subset \mathbb{R}^3$ be the line spanned by $e_1 + e_2 + e_3$. This is stable under S_3, i.e. it is a subrepresentation.

A representation is called <u>irreducible</u> if it contains no non-trivial (i.e. $\neq 0, V$) subrepresentation.

Little exercise: in the S_3 example above, convince yourself that we get an induced representation on $\mathbb{R}^3/U$, and that this representation is irreducible.

What's the point of this? It is a classical fact (<u>Maschke's theorem</u>) that if G is finite and we work with representations over a field of characteristic zero ($\mathbb{Q}, \mathbb{R}, \mathbb{C}, ..$), then every representation is (uniquely) a direct sum of irreducible representations. Hence, if we know all irreducible representations, we know all representations.

$\Rightarrow$ irreducible representations are the building blocks of (finite, linear) symmetry.

S_3 for example has 3 irreducible representations: the 2-dimensional we found above and two further 1-dimensional ones (which?).

In an abstract group, the elements can't see each other:

$$\underset{g}{\bullet} \qquad\qquad \underset{h}{\bullet}$$

"I'm so lonely" "I'm so lonely"

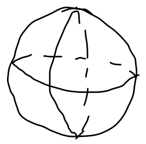
But actually, very often the symmetry transformations form some "geometric space" and can "see" each other. Consider for example the orientation preserving symmetries of the 2-sphere in $\mathbb{R}^3$. This is the special orthogonal group $SO(3)$. It contains for example the rotations

$$R_z(\Theta) := \begin{pmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Theta \in [0, 2\pi]$$

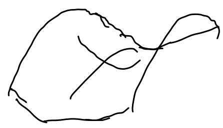
around the z-axis. They vary "continuously" with a parameter Θ and we can think of $R_z(\Theta)$ for "small" Θ as an "infinitesimal generator" L_z of rotations around the z-axis. This infinitesimal generator is a "tangent vector" to the "space" $SO(3)$.

But how is this a space? What is space?

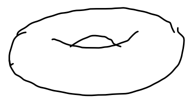
Our universe is probably not flat (like $\mathbb{R}^n$) but curved, e.g.



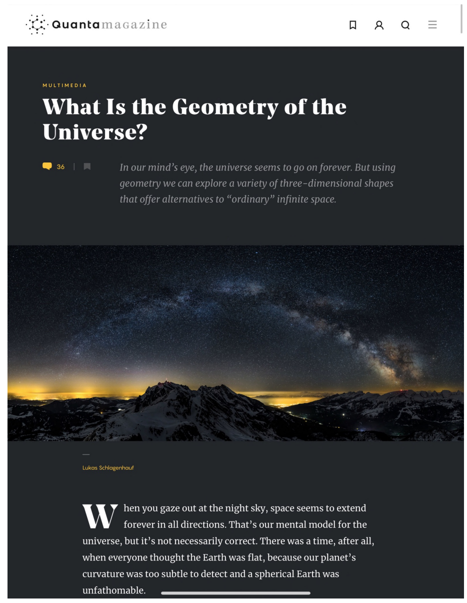
sphere



saddle



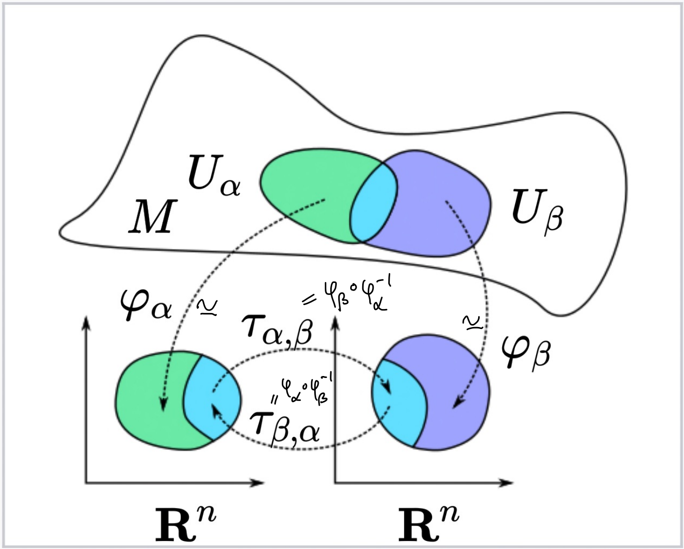
torus



Browser screenshot, check out

quantamagazine.org/what-is-the-geometry-of-the-universe-20200316/

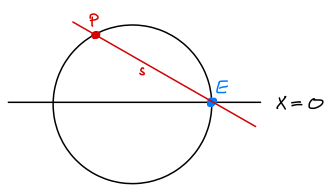
But locally (in a small neighbourhood of any point) it is flat, i.e. looks like (an open subset of) $\mathbb{R}^n$.
The precise notion of a space which "locally looks like $\mathbb{R}^n$" is that of a manifold. This is a (topological) space M which is glued together from open subsets of $\mathbb{R}^n$:



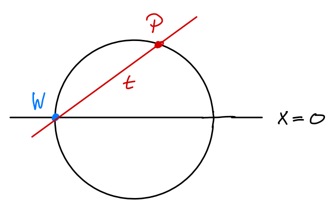
Two charts on a manifold, and their respective **transition map**

(picture from Wikipedia, author Stomatapoll, License CC BY-SA 3.0)

Here's an example: the circle $S^1 = \{ x \in \mathbb{R}^2 \mid \|x\|^2 = 1 \}$. How is this a manifold? It is patched together from two parts:



similarly:



$U_\alpha := S^1 \setminus \{E\} \xrightarrow[\simeq]{\varphi_\alpha} \mathbb{R}$

$\varphi_\alpha(P = (x,y)) =$ slope s of the line through P and E
$= \dfrac{y}{1+x}$

$U_\beta := S^1 \setminus \{W\} \xrightarrow[\simeq]{\varphi_\beta} \mathbb{R}$

$\varphi_\beta(P = (x,y)) =$ slope t of the line through P and W
$= \dfrac{y}{1-x}$

Transition function on the overlap is $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}, \quad s \mapsto \frac{1}{t}$.
This is a diffeomorphism (differentiable isomorphism). $\Rightarrow$ S^1 is a 1-dimensional manifold.

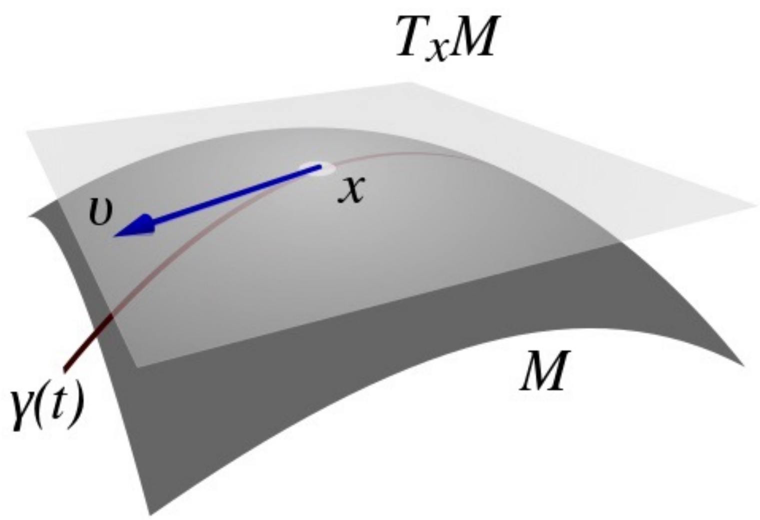
You can do a similar construction for the 2-sphere S^2, and more generally for S^n.

Let M be a manifold. We want to define <u>tangent directions</u> in a point $x \in M$.
Imagine, you sit in a car, drive with full speed on M, you drive through x, and then exactly in x, your car gets out of control and flies straight off M. That's a tangent vector.

More precisely: take a chart (U, φ) around x, $\varphi: U \xrightarrow{\simeq} V \subseteq \mathbb{R}^n$. Take a path

$$\gamma: (-1,1) \longrightarrow U \subseteq X$$

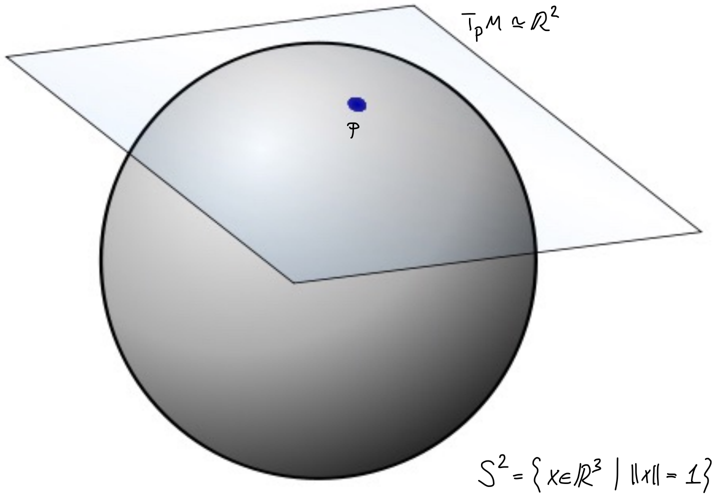
such that $\gamma(0) = x$ and such that $\varphi \circ \gamma: (-1,1) \longrightarrow \mathbb{R}^n$ is a differentiable map.
Then we can think of $D(\varphi\circ\gamma)(0) \in \mathbb{R}^n$ as a tangent vector in x on M.



(picture from Wikipedia)

Two distinct paths can yield the same tangent vector $\rightsquigarrow$ introduce an equivalence relation on paths $\rightsquigarrow$ get a vector space T_xM of equivalence classes (<u>tangent space</u> in x)
We have $T_xM \simeq \mathbb{R}^n$.

Here's an example for the 2-sphere (Wikipedia):



(picture from Wikipedia)

A (differentiable) map $f: X \to Y$ of manifolds induces a linear map

$$T_x f : T_x X \longrightarrow T_{f(x)} Y$$
$$\gamma \longmapsto f \circ \gamma \qquad \text{(transport path to } Y\text{)}$$

Back to the topic.

$\mathbb{R}^n$, and any open subset of $\mathbb{R}^n$, is obviously a manifold (there is nothing to patch). Hence,

$$End(\mathbb{R}^n) \simeq Mat_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$$

is a manifold (of dimension n^2). The determinant map

$$det : Mat_n(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto det\, A$$

is given by polynomials in the entries (coordinates) $\Rightarrow$ this is a continuous map, hence

$$GL_n(\mathbb{R}) = det^{-1}(\underbrace{\mathbb{R}\setminus\{0\}}_{\text{an open subset of } \mathbb{R}})$$

is an open subset of $Mat_n(\mathbb{R})$ $\Rightarrow$ it is a (sub-) manifold and

$$\begin{array}{ccc} T_x(GL_n(\mathbb{R})) & \overset{\text{naturally}}{\simeq} & Mat_n(\mathbb{R}) \\ \wr\wr & & \wr\wr \\ \mathbb{R}^{n^2} & \simeq & \mathbb{R}^{n^2} \end{array}$$

Now, what about $SO(3) = \{A \in Mat_3(\mathbb{R}) \mid A \cdot A^t = I,\ det(A) = 1\}$?

related to implicit function theorem.

<u>Preimage Theorem</u>: Let $f: X \to Y$ be a differentiable map of manifolds and let $y \in Y$ be a <u>regular value</u> (meaning that $T_x f : T_x X \to T_y$ is surjective $\forall x \in f^{-1}(y)$). Then $f^{-1}(y) \subseteq X$ is a submanifold.

Apply this to the map

$$f: Mat_n(\mathbb{R}) \longrightarrow Mat_n(\mathbb{R})$$
$$A \longmapsto A \cdot A^t$$

This map is differentiable and one can show that $I \in Mat_n(\mathbb{R})$ is a regular value.

$$\Rightarrow O(n) = \{A \in Mat_n(\mathbb{R}) \mid A \cdot A^t = I\} = f^{-1}(Id)$$

is a submanifold.

Note $A \cdot A^t = I \Rightarrow \det(A)^2 = 1 \Rightarrow \det A = \pm 1$. Hence,

$$SO(n) = \left(\det|_{O(n)}\right)^{-1}\big(\underbrace{\mathbb{R}_{>0}}_{\text{open subset of } \mathbb{R}}\big)$$

is an open subset of $O(n) \Rightarrow$ it is a manifold. It's dimension is $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

So, e.g. $\dim SO(3) = 3$. Hence, $\dim T_{Id}(SO(3)) = 3$, i.e. we have 3 tangent directions.

$\uparrow$ Identity

The three tangent directions are precisely the infinitesimal generators L_x, L_y, L_z of rotation around the x, y, z-axis, respectively.

Now, the tangent space is in this case not just a vector space, it has more structure. Since $SO(n)$ is an open submanifold of $O(n)$ and $O(n)$ is a closed submanifold of $GL_n(\mathbb{R})$, we naturally have an embedding

$$so(n) := T_{Id}(SO(n)) = T_{Id}(O(n)) \hookrightarrow T_{Id}(GL(n)) =: gl(n) \simeq Mat_n(\mathbb{R})$$

Under this identification, $so(n) = \{$skew-symmetric matrices, i.e. $A^t = -A\}$

For $so(3)$ one can identify

$$L_{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_{\mathbf{z}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The <u>commutator</u> of two matrices $A, B \in Mat_n(\mathbb{R})$ is

$$[A, B] := AB - BA \qquad ([A,B] = 0 \text{ iff } A, B \text{ commute})$$

One can check that under the above embedding $so(n)$ is stable under taking commutators (i.e. if $A, B \in so(n)$ then $[A,B] \in so(n)$). You can check that for $so(3)$:

$$[L_x, L_y] = L_z, \quad [L_z, L_x] = L_y, \quad [L_y, L_z] = L_x$$

These commutator relations between the infinitesimal generators of rotation is of fundamental importance.

In general: a group which is at the same time a manifold is called a <u>Lie group</u>.
If G is a Lie group, one can always define a bracket $[\cdot,\cdot]$ on the tangent space $Lie(G) := T_{Id}G$ satisfying similar properties as the commutator.
Such a structure is called a <u>Lie algebra</u>.

The Lie algebra of a Lie group is an "<u>infinitesimal residue</u>" of the Lie group.
It is so important because it is much simpler (vector space + extra structure) but still sees a lot of G.

For example, if $\rho: G \longrightarrow GL(V)$ is a representation of a Lie group, then its differential gives a linear map

$$T_{id}\rho : Lie(G) = T_{id}G \longrightarrow T_{id}\, GL(V) = \mathfrak{gl}(V)$$

satisfying

$$(T_{id}\rho)([A,B]) = [T_{id}(\rho)(A), T_{id}(\rho)(B)],$$

i.e. $T_{id}\rho$ is a <u>representation</u> of the Lie algebra $Lie(G)$!

There is also a partial converse to this.

Lie algebras and their representations are thus a key tool in studying Lie groups and their representations. This is the motivation for this course.

A nice feature of Lie theory is that often things are controlled by discrete combinatorial data $\rightsquigarrow$ allows explicit calculations and computer experiments.

Curious? Then follow the course! We will only be concerned with Lie algebras, no manifolds, so <u>don't worry if you didn't understand all details</u>.