Introduction to categorical thinking and categorification

a.k.a.

Introduction to tensor categories

Very very very early version (without tensor categories)—comments welcome!

Ulrich Thiel

Email address: `thiel@mathematik.uni-kl.de`

Web address: `https://ulthiel.com/math`

University of Kaiserslautern, Department of Mathematics, 67653 Kaiserslautern, Germany

v0.1 (Feb 2021)

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Introduction

You all know what a ring is: it's a set with an addition and a multiplication. But the poor elements of a set just can't talk to each other. Isn't this depressing? So, let's lift the concept of a set up a level and allow elements (objects) to talk to each other via morphisms. What we get is called a category and what we just did was "categorifying" a mathematical concept. Let's assume we also have a multiplication on our set and that it has a unit element, i.e. we have a monoid. Can we categorify this concept, too? Of course: we not just multiply objects but also morphisms and we need a unit object. What we get is called a monoidal category. By now you can believe that lifting the concept of addition up a level will work like charm as well, and when we have all that in a compatible manner we call this a tensor category. So, a tensor category is the categorification of the concept of a ring. A simple (but boring) example is the category of vector spaces with tensor product and direct sum. There are many more examples from all over mathematics, mathematical physics, and even computer science. The fun thing is that on the categorical level things happen that you just can't see a level downstairs. This is the basic theme of categorification. The structure of tensor categories is rich and fascinating, they are subject of extensive research.

The goal of this course is to expose you to categorical thinking and the general idea of categorification. Tensor categories make an excellent topic for this. I have selected some examples, constructions, and results that I find interesting and hope you enjoy as well. I will not assume you know about categories already and will tell you what's necessary to know without getting lost too much in abstract nonsense. I believe that familiarity with basic algebraic structures should be sufficient—but the more you have seen the better it will be. Please send feedback if anything is unclear (I may want to publish these notes eventually and any feedback will be helpful). My introduction above was quite sloppy on purpose but the main text will be precise.

The notes are at a very early stage and cover so far only some basic category theory. My idea was to prepare you for the excellent recent book on tensor categories by Etingof, Gelaki, Nikshych, and Ostrik [4]. Once I have given a few iterations of this course, I will try to cover basics of tensor categories here as well.

Acknowledgments

I would like to thank all of the following people for providing feedback on my notes. Cedric Brendel, Alexander Dinges, Markus Kurtz, Tamara Linke, Helena Petri, Adrian Rettich, Erec Thorn (all from my 2020/2021 course).

CHAPTER 1

Categories

Whenever you introduce a mathematical structure (like groups, vector spaces, topological spaces) you also want to be able to relate objects having such a structure. Especially, you want to be able to make precise what it means for two objects being "structurally equal". Often, there is a natural notion of "structure preserving maps" and you consider two objects as structurally equal when you have such maps in both directions which are pairwise inverses of each other. The technical term for a structure preserving map is **homomorphism** (coming from the Greek *homos* meaning "similar" and *morphe* meaning "form" or "shape") and the technical term for an invertible homomorphism as above is **isomorphism** (from Greek *iso* meaning "equal"). Most people just say **morphism** instead of the longer word homomorphism. You should now pause for a minute and recall what morphisms and isomorphisms are for groups, vector spaces, and topological spaces.

I'm sure you will immediately agree that the concept of "objects" and "morphisms" is very general and occurs everywhere in mathematics (and beyond). The formal mathematical stage to deal with objects and morphisms is that of a **category**. Even though this concept is very natural, it was introduced only in 1945 by Samuel Eilenberg and Saunders MacLane [3]. Their working ground at this time was *algebraic topology*. This is the study of topological spaces by algebraic means, so you are connecting two completely different worlds: the topological category and an algebraic category like the category of groups. Even though category theory is too general to tell you how to do this *explicitly*, it will still guide you what to look for and exposes *general principles* of such a connection. This is extremely powerful. Category theory is not just a language as some people say; it is a way of thinking and of approaching mathematical problems. It will change you forever.

Are you curious? Then let's go. I emphasize that algebraic topology will not play a role here—it is simply one motivating context and an excellent illustration of the power of these concepts. Category theory is a vast subject with infinitely many applications, even going deep into philosophy. I will only touch some basics I will need in the course. If you want to know more, there's the classic reference by MacLane [8] and, e.g., [1], [11], and [2].[1]

1.1. Definition and basic examples

A category has objects and morphisms, each morphism has a source and a target object, and whenever you have two morphisms such that the target of one is

[1] I personally find the book by MacLane a bit dry but please ignore my opinion and see yourself. I actually never read a category theory book cover to cover; I picked up things on the fly when I had a problem or application in mind. Actually, for the very basics I really simply recommend the Wikipedia articles!

the source of the other, you can compose them; composition should be associative and have an identity. Let's write this down formally.

DEFINITION 1.1.1. A **category** $\mathcal{C}$ consists of:

- a collection $\mathcal{C}_0$ of **objects**,
- a collection $\mathcal{C}_1$ of **morphisms**,
- maps $s\colon \mathcal{C}_1 \to \mathcal{C}_0$ giving the **source** (or **domain**) and $t\colon \mathcal{C}_1 \to \mathcal{C}_0$ giving the **target** (or **codomain**) of morphisms,
- a map
$$\circ\colon \{(f,g) \in \mathcal{C}_1 \times \mathcal{C}_1 \mid t(f) = s(g)\} \to \mathcal{C}_1 \tag{1.1}$$
giving the **composition** of pairs of **composable** morphisms, and this map has to satisfy the following properties:
  (1) source and target are respected, i.e.
  $$s(g \circ f) = s(f) \quad \text{and} \quad t(g \circ f) = t(g)\ , \tag{1.2}$$
  (2) associativity, i.e.
  $$(h \circ g) \circ f = h \circ (g \circ f) \tag{1.3}$$
  whenever $t(f) = s(g)$ and $t(g) = s(h)$,
  (3) existence of an **identity**, i.e. for each $X \in \mathcal{C}_0$ there is $\mathrm{id}_X \in \mathcal{C}_1$ with $s(\mathrm{id}_X) = X = t(\mathrm{id}_X)$ and
  $$f \circ \mathrm{id}_X = f \quad \text{and} \quad \mathrm{id}_X \circ g = g \tag{1.4}$$
  for any $f \in \mathcal{C}_1$ with $s(f) = X$ and any $g \in \mathcal{C}_1$ with $t(g) = X$.

We simply write $X \in \mathcal{C}$ instead of $X \in \mathcal{C}_0$. Also, we often write $\mathrm{Ob}_\mathcal{C}$ instead of $\mathcal{C}_0$ and $\mathrm{Mor}_\mathcal{C}$ instead of $\mathcal{C}_1$. The word morphism is short for **homomorphism** and it is more common to write $\mathrm{Hom}_\mathcal{C}$ instead of $\mathrm{Mor}_\mathcal{C}$. We write $f\colon X \to Y$ to indicate that f is a morphism with source X and target Y. By $\mathrm{Hom}_\mathcal{C}(X,Y)$ we denote the collection of all such morphisms. For any triple X, Y, Z of objects the composition (1.1) gives a map
$$\circ\colon\ \mathrm{Hom}_\mathcal{C}(X,Y) \times \mathrm{Hom}_\mathcal{C}(Y,Z) \to \mathrm{Hom}_\mathcal{C}(X,Z)\ . \tag{1.5}$$
In practice, it is often more convenient to specify a category by specifying a morphism collection $\mathrm{Hom}_\mathcal{C}(X,Y)$ for all pairs of objects and a composition map as in (1.5) for all triples of objects subject to associativity and identity. Then one takes $\mathrm{Hom}_\mathcal{C}$ to be the disjoint union of the $\mathrm{Hom}_\mathcal{C}(X,Y)$, and the source and target maps are the obvious ones. I call this a "local" definition in contrast to a "global" definition. As in group theory you can easily see that there is a *unique* identity morphism for each object: if there is another identity id'_X, then
$$\mathrm{id}'_X = \mathrm{id}'_X \circ \mathrm{id}_X = \mathrm{id}_X \tag{1.6}$$
by the property of the identity. The notation 1_X for id_X is also very common.

Let's look at some examples to give some life to this abstract nonsense.

EXAMPLE 1.1.2. The prime example of a category is the **category of sets**, denoted Set: the objects are sets, the morphisms are maps, the composition is the usual composition of maps, and the identity is the identity map. Many mathematical structures you are already familiar with are sets with extra structure and morphisms

Category	Objects	Morphisms
Set	sets	maps
Mon	monoids	monoid morphisms
Grp	groups	group morphisms
Ab	abelian groups	group morphisms
K-Vec	K-vector spaces	K-linear maps
Ring	rings	ring morphisms
CRing	commutative rings	ring morphisms
R-Mod	(left) R-modules	R-linear maps
R-Alg	R-algebras	R-algebra morphisms
Top	topological spaces	continuous maps
Man	smooth manifolds	C^∞-maps
Ban	Banach spaces	bounded operators

TABLE 1.1. Basic examples of categories. I am not mentioning the composition and identity because it's always the set-theoretic one. Recall that a **monoid** is a set equipped with a binary operation that is associative and that has a neutral element (so, a group is a monoid in which every element is invertible). A monoid morphism is a map that is compatible with the operation (like a group morphism). A **ring** in this course is always assumed to be associative with unit, and ring morphisms are always assumed to preserve the unit. An **R-module** over a ring R is defined like a vector space—just over a ring. For a commutative ring R, an **R-algebra** is a ring A equipped with an R-module structure that is compatible with the multiplication in A, i.e. $r(aa') = (ra)a' = a(ra')$; and an R-algebra morphism is a ring morphism $f\colon A \to A'$ with $f(ra) = rf(a)$.

are maps of sets preserving the extra structure. As a rule of thumb, this always gives you a category. Table 1.1 lists some examples.

EXERCISE 1.1.3. What's your favorite example of a category?

All examples of categories we looked at so far were sets with extra structure. This story gets a bit boring. The main point of categories is that they are completely abstract and general. Look at the definition: objects don't need to have an underlying set and morphisms don't need to be maps of sets. Let's look at some more *abstract* examples.

EXAMPLE 1.1.4. We first go easy and consider again sets as objects. But as morphisms between two sets X and Y we now consider subsets $R \subseteq X \times Y$, i.e. **relations** between X and Y. Why does this form a category? First, part of the data of a category is the composition. What is the composition of two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$? We define it as

$$S \circ R \coloneqq \{(x,z) \in X \times Z \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in Y\} . \tag{1.7}$$

Convince yourself that this is associative. This composition has an identity, namely the trivial relation

$$1_X \coloneqq \{(x,x) \mid x \in X\} . \tag{1.8}$$

Hence, we get the category Rel of relations. While objects still have an underlying set, morphisms are not maps of sets anymore (note that a map is a *special* relation). Hence, things like $f(x)$ for a morphism f in this category don't make any sense!

EXAMPLE 1.1.5. We still go easy and consider a group G and fix a field K. Recall that the general linear group $\mathrm{GL}(V)$ of a K-vector space V is the group of all invertible linear maps $V \to V$. After choosing a basis of V, this group is isomorphic to $\mathrm{GL}_n(K)$ where n is the dimension of V. A **representation** of G over K is a group morphism $\rho\colon G \to \mathrm{GL}(V)$ from G into the general linear group of a K-vector space V. So, for every $g \in G$ you get an automorphism $\rho(g)$ on V, i.e. a "linear symmetry" of V and these symmetries obey the relations in G. This is how you should think about representations. There's an obvious notion of morphism between representations $\rho\colon G \to \mathrm{GL}(V)$ and $\rho'\colon G \to \mathrm{GL}(V')$, namely a linear map $f\colon V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \rho'(g)} \\ V & \xrightarrow{f} & V' \end{array} \tag{1.9}$$

commutes for any $g \in G$. Maybe you have never heard of commutative diagrams: a **diagram** is simply a collection of objects in some category connected by some morphisms; and the diagram is **commutative** when all paths in the diagram with the same start and end point lead to the same result after composition of all the morphisms that make up the path. The commutativity of (1.9) simply means that

$$f \circ \rho(g) = \rho'(g) \circ f \ . \tag{1.10}$$

You can convince yourself that with the usual composition of maps you get a category that we will denote by $\mathsf{Rep}_K(G)$. While morphisms in this category are still maps, objects are also maps and things like $x \in \rho$ don't really make sense.

EXAMPLE 1.1.6. Now, we go completely abstract. A **quiver**[2] Q consists of a set Q_0 of **vertices**, a set Q_1 of **arrows**, and maps $s, t\colon Q_1 \to Q_0$ assigning to each arrow $a \in Q_1$ its **source** $s(a)$ and **target** $t(a)$. So, a quiver is basically a directed graph but we also allow loops and multiple parallel edges which is usually excluded in the definition of directed graphs. Here's an example of a quiver:

$$\bullet \rightrightarrows \bullet \tag{1.11}$$

A **morphism** $f\colon Q \to Q'$ between quivers consists of maps $f_0\colon Q_0 \to Q_0'$ and $f_1\colon Q_1 \to Q_1'$ such that

$$f_0 \circ s = s' \circ f_1 \quad \text{and} \quad f_0 \circ t = t' \circ f_1 \ . \tag{1.12}$$

So, a morphism simply takes vertices to vertices and arrows to arrows while respecting source and target of the arrows. We have an obvious composition of quiver morphisms and an identity morphism on each quiver, hence quivers form a category Quiv. Now, things like $x \in Q$ and $f(x)$ don't really make sense.

You have probably noticed that the definition of a quiver was already part of our Definition 1.1.1 of a category! Any category *is* a quiver—only with the extra datum of a composition and with a loop at each vertex coming from the identity. We denote

[2] A "quiver" is an archer's portable case for holding arrows. Pause a minute to admire this beautiful terminology.

the **underlying quiver** of a category $\mathcal{C}$ by $Q(\mathcal{C})$. Viewing categories as (upgraded) quivers, forces you to abandon thinking about objects having an underlying set and morphisms being actual maps. If your objects and morphisms happen to be of set-theoretic nature, you should of course use this! But when working with general categories, forget it!

EXERCISE 1.1.7. Show that any group G can naturally be considered as a category with one object.

EXERCISE 1.1.8. Show that any partially ordered set can naturally be considered as a category.

Add examples of empty category and discrete categories.

1.2. Subcategories

Recall the category Set of sets. Of course we can also consider *finite* sets with maps between them and get a category that we denote by set, i.e. with lower case starting letter. The objects are just special objects of its big brother Set and the morphisms are exactly the same. You may already guess that set is a *sub*category of Set, and such things occur all the time so let's make this notion precise.

DEFINITION 1.2.1. A **subcategory** $\mathcal{C}$ of a category $\mathcal{C}'$ consists of:

- a subcollection $\mathcal{C}_0 \subseteq \mathcal{C}'_0$,
- a subcollection $\mathcal{C}_1 \subseteq \mathcal{C}'_1$,

such that

(1) for every $f \in \mathcal{C}_1$ both the source and target of f are in $\mathcal{C}_0$,
(2) $\mathcal{C}_1$ is closed under composition in $\mathcal{C}'_1$,
(3) $\mathrm{id}_X \in \mathcal{C}_1$ for all $X \in \mathcal{C}_0$.

The subcategory is called **full** if $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{C}'}(X,Y)$ for all $X, Y \in \mathcal{C}$.

You can easily convince yourself that $\mathcal{C}$ with the composition and source and target maps from $\mathcal{C}'$ forms itself a category. Full subcategories are easy to specify: you just need to specify a subcollection $\mathcal{C}_0$ of objects, and then you take $\mathcal{C}_1$ to be the subcollection of morphisms whose source and target is in $\mathcal{C}_0$; all properties are then automatically satisfied. Let's look at some examples.

EXAMPLE 1.2.2. We already noticed that set is a full subcategory of Set. Similarly, grp is the full subcategory of Grp of finite groups, ab is the full subcategory of Ab of finite abelian groups (this is at the same time also a full subcategory of grp and of Grp), K-vec is the full subcategory of K-Vec of finite-dimensional vector spaces, and R-mod is the full subcategory of R-Mod of finitely generated R-modules. If we have an algebra A over a field K, e.g. the polynomial ring $K[X]$, we often consider the full subcategory A-fdmod of A-modules which are finite-dimensional over K. The field is dropped from the notation but usually this doesn't cause confusion. The category A-fdmod is a full subcategory of A-mod, and we have equality if A is finite-dimensional over K. Along this line, $\mathsf{rep}_K(G)$ is the full subcategory of $\mathsf{Rep}_K(G)$ of finite-dimensional representations of G.

Please note that in the literature you will find many different notations for the categories in Table 1.1 and prominent subcategories. For example, in [4] the category of all K-vector spaces is K-**Vec** and the finite-dimensional category is K-Vec. You should always check the conventions.

EXAMPLE 1.2.3. An example of a non-full subcategory is the subcategory Set of the relation category Rel introduced in Example 1.1.4 (recall that maps are special relations). Another example is the category Man^p of C^p-manifolds with C^p-maps. Then Man^{p+1} is a subcategory of Man^p and you may recall from analysis that there are C^p-maps on the real line which are not C^{p+1}, so this is not a full subcategory.

1.3. Special morphisms

Let's talk about morphisms in a general category $\mathcal{C}$. The following notion should not come as a surprise.

DEFINITION 1.3.1. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is an **isomorphism** if there is a morphism $g\colon Y \to X$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

You can easily see that if there is such a g it is unique and we call it the **inverse** of f: suppose there is another such g', then

$$g' = g' \circ \mathrm{id}_Y = g' \circ (f \circ g) = (g' \circ f) \circ g = \mathrm{id}_X \circ g = g \ . \tag{1.13}$$

What are the isomorphisms in Set? Of course, they are the *bijective* maps. Let's recall that a map $f\colon X \to Y$ is *bi*jective if it is *in*jective, i.e. $f(x) = f(x')$ implies $x = x'$, and *sur*jective, i.e. for any $y \in Y$ there is an $x \in X$ such that $f(x) = y$. Do you see a problem with these notions? Yes, *they are not categorical!* We are working with elements $x \in X$ and values $f(x)$, which we learned don't make any sense in a general category! Hence, we cannot define isomorphisms in a general category as being injective and surjective morphisms. Equation 1.13 on the other hand is purely categorical and the correct general concept.

Of course, if you have a category where the morphisms are actual maps between sets (all examples in Table 1.1) you can still talk about the non-categorical notion of bijectivity. You can easily convince yourself that in any such category an isomorphism will always be bijective (the inverse morphism gives a set-theoretic inverse making the map bijective). If the set-theoretic inverse then also happens to be a morphism in your category, then you have an isomorphism. Sometimes, this is in fact true: in the category of groups, rings, vector spaces, and modules the isomorphisms are precisely the bijective morphisms (convince yourself of this again). But this is simply a fortunate coincidence! Do you know an example where this is not true, where a bijective morphism is not an isomorphism in the category? Yes, the category Top of topological spaces! There are standard examples of bijective continuous maps whose set-theoretic inverse is not continuous, hence not an isomorphism in the category. But now that you can think categorically, this should not even surprise you and you would have never considered bijective continuous maps as isomorphisms of topological spaces! By the way, the isomorphisms in Top have a special name (not that we would need one): they are called **homeomorphisms**. Similarly, isomorphisms in Man are called **diffeomorphisms**.

EXERCISE 1.3.2. Give an example of a bijective continuous map which is not an isomorphism in Top.

A morphism from an object X to itself is called an **endomorphism** (from Greek *endo* meaning "within"). The set of endomorphisms on X is denoted by $\mathrm{End}_{\mathcal{C}}(X)$. Convince yourself that $\mathrm{End}_{\mathcal{C}}(X)$ is a monoid with the composition of morphisms as multiplication and id_X as unit element. An endomorphism which is also an isomorphism is called an **automorphism**. We denote by $\mathrm{Aut}_{\mathcal{C}}(X)$ the set of automorphisms of X. This is exactly the group of units in the monoid $\mathrm{End}_{\mathcal{C}}(X)$.

Let's come back to the non-categorical notions of injective and surjective. Let's not give up on them yet. Can we somehow re-formulate them categorically? Yes, we can!

LEMMA 1.3.3. *Let $f\colon X \to Y$ be a map of sets.*

(1) *f is injective if and only if whenever $f \circ g_1 = f \circ g_2$ for any two maps $g_1, g_2\colon Z \to X$, then already $g_1 = g_2$.*
(2) *f is surjective if and only if whenever $g_1 \circ f = g_2 \circ f$ for any two maps $g_1, g_2\colon Y \to Z$, then already $g_1 = g_2$.*

PROOF. I just consider the first claim, the second is proven similarly. Suppose that f is injective. Then $f(g_1(x)) = f(g_2(x))$ implies $g_1(x) = g_2(x)$ for any $x \in X$, so $g_1 = g_2$. Conversely, suppose the other property holds. Assume that $f(x) = f(x')$ for some $x, x' \in X$. Let $\{*\}$ be a one-element set and define maps $g_1, g_2\colon \{*\} \to X$ with $g_1(*) = x$, $g_2(*) = x'$. Then $f \circ g_1 = f \circ g_2$, so $g_1 = g_2$, implying that $x = x'$, so f is injective. □

Note that in these re-formulations we just use morphisms and equality of morphisms, no $x \in X$ and no $f(x)$. So, this is purely categorical and we can consider this concept in any category. Morphisms satisfying the categorical condition in 1 are called **monomorphisms**, and morphisms satisfying the categorical condition in 2 are called **epimorphisms**. As we have just seen, in the category Set the monomorphisms are precisely the injective maps and the epimorphisms are precisely the surjective maps. In any category where morphisms are actual maps of sets (all examples in Table 1.1), it is clear that any injective morphism is a monomorphism and any surjective morphism is an epimorphism. However, the converse may not be true and these categorical notions are in fact quite tricky, you need to be very careful!

EXAMPLE 1.3.4. Consider the category Ring of rings. We claim that a monomorphism $f\colon R \to S$ is injective (so that monomorphisms in Ring are precisely the injective ring morphisms). Suppose it is not. Then there are distinct elements $r, r' \in R$ with $f(r) = f(r')$. Consider the ring morphisms $g_1, g_2\colon \mathbb{Z}[T] \to R$ with $g_1(T) = r$ and $g_2(T) = r'$. Then $f \circ g_1 = f \circ g_2$, so $g_1 = g_2$ since f is a monomorphism but this is a contradiction. Note that the reason this works is that we have the polynomial ring $\mathbb{Z}[T]$ which has the nice property that we can specify a ring morphism *out* of it by just specifying where T maps to. This is the ring-analogue of the one-element set $\{*\}$ that we used in the proof of Lemma 1.3.3.

What about epimorphisms in Ring? Are they precisely the surjective ring morphisms? I have to disappoint you! Consider the beautiful ring morphism $f\colon \mathbb{Z} \to \mathbb{Q}$ (there is just one). This is clearly injective and not surjective. I claim it is an epimorphism! In fact, let $g_1, g_2\colon \mathbb{Q} \to R$ be ring morphisms with $g_1 \circ f = g_2 \circ f$, so $g_1(n) = g_2(n)$ for all $n \in \mathbb{Z} \subset \mathbb{Q}$. Since g_1, g_2 are ring morphisms we have

$$g_1\left(\frac{n}{m}\right) = g_1(n \cdot m^{-1}) = g_1(n) \cdot g_1(m)^{-1} = g_2(n) \cdot g_2(m)^{-1} = g_2\left(\frac{n}{m}\right) \;,$$

so $g_1 = g_2$ and f is an epimorphism. Note that the reason this works is that a ring morphism from $\mathbb{Q}$ is already uniquely determined on $\mathbb{Z}$. So, what are the epimorphisms in Ring then? This is actually an open problem![3]

Example 1.3.5. In the category Grp of groups, monomorphisms are again injective. Namely, if $f\colon G \to H$ is a monomorphism, consider the embedding morphism $g_1\colon \operatorname{Ker}(f) \to G$ and the morphism $g_2\colon \operatorname{Ker}(f) \to G$ mapping any $g \in \operatorname{Ker}(f)$ to $1 \in G$. Then $f \circ g_1 = f \circ g_2$, so $g_1 = g_2$ because f is a monomorphism, but this means that $\operatorname{Ker}(f) = 1$, so f is injective. As before, we needed a special object—in this case the kernel—to prove this.

What about epimorphisms in Grp? In contrast to the situation in Ring, it is in Grp in fact true that epimorphisms are surjective—but this is not so easy to prove! The claim follows from the general amalgamation theorem due to Schreier [12] (from 1927). There is an easier direct proof due to Linderholm [7] (from 1970) that I will recite here. Let $f\colon G \to H$ be an epimorphism of groups. We need to show that $\operatorname{Im}(f) = H$. Let $X \coloneqq H/\operatorname{Im}(f)$ be the set of all right cosets of the subgroup $\operatorname{Im}(f)$ in H. Let ∞ be some symbol, not a right coset of $\operatorname{Im}(f)$ in H, and define $Y \coloneqq X \cup \{\infty\}$. Let S be the symmetric group on the set Y. Right multiplication on X with elements from H induces an embedding $g_1\colon H \to S$ of H into S as permutations that fix ∞. Let $\sigma \in S$ be the permutation that exchanges the coset $\operatorname{Im}(f)$ with ∞ and fixes everything else. Let $g_2\colon H \to S$ be the morphism defined by $g_2(h) \coloneqq \sigma^{-1} \circ g_1(h) \circ \sigma$. If $h \in \operatorname{Im}(f)$, then by definition $g_1(h)$ fixes the coset $\operatorname{Im}(f)$ and the element ∞, hence it commutes with σ. But this means $g_1(h) = g_2(h)$. Hence, $g_1 \circ f = g_2 \circ f$, so $g_1 = g_2$ because f is an epimorphism. But this means that $g_1(h)$ and σ commute for any $h \in H$. This requires that $g_1(h)$ must fix the coset $\operatorname{Im}(f)$. But this in turn requires that $h \in \operatorname{Im}(f)$. We thus must have $H = \operatorname{Im}(f)$, so f is surjective as claimed.

Notice the following: after having established this property of epimorphisms in Grp it is a priori *not* clear that the same also holds in the subcategory grp of *finite* groups (think about it). But when you look into the proof you see that if H is finite, so is the group S and the whole proof stays *inside* the subcategory grp, hence also in grp every epimorphism is surjective—we are lucky.

Could give example for R-modules as well.

You see that when you generalize some familiar notions of maps to morphisms in categories—assuming this is even possible—things can become quite complicated. We will not dig further into this topic here.

1.4. Set-theoretic issues

I was cheating the whole time, right from Definition 1.1.1 on! Did you notice it? In the definition I was talking about a "collection" of objects and of morphisms. What is a "collection"? This is just another word for "set" somehow, so why didn't I just say "set"?

The problem is set theory. Consider the category Set of sets, so $\mathrm{Ob}_{\mathsf{Set}}$ is the "collection" of all sets. Now you may remember from set theory that you cannot form the set of all sets. One particular problem that arises is **Russell's paradox** discovered in 1901. Suppose that $\mathrm{Ob}_{\mathsf{Set}}$ is a set. Then we can form the set $R \coloneqq$

[3] Any localization morphism $R \to R[S^{-1}]$ is an epimorphism for the same reason $\mathbb{Z} \to \mathbb{Q}$ is. But there are ring epimorphisms which are neither surjective nor localization morphisms and it is very difficult to classify the epimorphisms.

$\{x \in \mathrm{Ob}_{\mathsf{Set}} \mid x \notin x\}$ of all sets not containing themselves. Since R is a set, we can ask whether $R \in R$ or $R \notin R$. But by construction $R \in R$ implies $R \notin R$, and conversely $R \notin R$ implies $R \in R$. So, we end up in a contradiction! And maybe you know from logic that once you have a contradiction in a logical system, you can prove *any* statement to be true, so this is very bad!

I will now tell you how to save the ship. I will say a bit more than is actually necessary in practice, so your head might start spinning a bit, but I feel it is important to have heard about this once. In the end, we'll be very pragmatic about these issues.

The conclusion of Russell's paradox is that we need to reconsider under which circumstances and in which ways we are allowed to build sets. This is called **axiomatic set theory**, in contrast to **naive set theory** where one doesn't care about anything and arrives at problems like the above. The earliest and still most widely used axiomatic system is due to Zermelo and Fraenkel (**ZF** for short), developed between 1908 and 1922. We will not need the precise axioms here. What you just need to know is that it is always fine to build from a *set* $\mathcal{U}$ and a property ϕ the *sub*set $\{x \in \mathcal{U} \mid \phi(x)\}$ of all elements of $\mathcal{U}$ satisfying this property. This is an axiom of ZF, called **restricted comprehension**. Part of the other axioms also tell you that there is an empty set $\emptyset$. So, there exists at least one set; and then you can start building a 1-element set $\{\emptyset\}$, a 2-element set $\{\emptyset, \{\emptyset\}\}$, and so on; and by taking unions, intersections, power sets, etc. you can build more and more complicated sets—all the way up to the real numbers and beyond. An axiom which is not part of ZF is the **axiom of choice** which states that the Cartesian product $\prod_{i \in I} X_i$ of a family of non-empty sets X_i indexed by a set I is non-empty, so that it is possible to make a choice $(x_i)_{i \in I}$ of an element $x_i \in X_i$ for each $i \in I$ at once. You may find it strange that this "obvious" statement is an axiom but one can show that it is in fact *independent* of the axioms of ZF. The extension of ZF adding the axiom of choice is called **ZFC**. Some people decide not to assume choice at all or point out explicitly whenever they have to assume it. But I find life *with* choice much more beautiful and will therefore *always* assume this.[4]

Restricted comprehension in axiomatic set theory is in contrast to **unrestricted comprehension** in naive set theory where we build sets out of nowhere, i.e. by not necessarily specifying a base set $\mathcal{U}$ as above. Precisely this is at the core of Russell's paradox: we would define the set of sets as $\{x \mid \}$ by using unrestricted comprehension. In axiomatic set theory, this construction is not allowed, hence the set of sets doesn't exist, and there is no Russell's paradox—problem solved![5]

[4]For example, basically all of commutative algebra would not work without the axiom of choice: one can show that choice is equivalent to the statement that every commutative ring has a maximal ideal, and what can you do in commutative algebra without this property?

[5]The fact that ZF erases this one particular issue (and some similar) does not mean that it is free of any inconsistencies! But now you may wonder why I'm saying this. Isn't the point of an axiomatic system that it is consistent? Well, here comes Gödel with his **incompleteness theorems** from 1931. If you have a consistent axiom system that is sophisticated enough to model the natural numbers then: 1) it is incomplete (there are true statements which you cannot prove in the system); 2) you cannot prove the consistency of the system within the system. Hence, we don't know whether ZF is consistent and we cannot prove it! We only *believe* it is consistent! And you always thought mathematics was rock solid!?

But still you can imagine a "collection" $\mathrm{Ob}_{\mathsf{Set}}$ of all sets, even when it is not a set in the axiomatic sense. Is there any way to deal with this? Yes! You just have to extend your axiom system to include a new type called **class**. Every set is a class (so you have a **hierarchy** of types), you can do operations with classes like you do with sets, and there is a **class comprehension** axiom which asserts that you can build the *class* of all *sets* satisfying some property. In particular, you can form the *class* $\mathrm{Ob}_{\mathsf{Set}}$ of all *sets*! If you are confused and think we were cheating by just introducing another word for "set", let me assure you that—similar as before—we are not allowed to form the class of all classes! So, there is no Russell's paradox. The class $\mathrm{Ob}_{\mathsf{Set}}$ is really a **proper class**, meaning it is not in the lower hierarchy of sets (because then we would have Russell's paradox again).

A formal class extension of ZFC was developed by Neumann, Bernays, and Gödel (**NBG** for short) between 1925 and 1940. A more flexible and powerful class extension of ZFC (even of NBG) is due to Morse and Kelley (**MK** for short) from about 1950. Maybe you wonder already that when you can define a type extension allowing you to form a class of sets, can't you continue this and introduce a further type into the hierarchy (let's call it **conglomerate**) so that we can form the conglomerate of classes? Yes, sure you can do this. And of course you can continue this process. You will probably run out of words for the new types in your hierarchy. There is an axiomatic system due to Tarski and Grothendieck (**TG** for short) which extends MK and allows very flexible infinite chains of such hierarchies based on the notion of **Grothendieck universes**.

We will not need any details of these set-theoretic foundations here because this topic is known to cause health problems. If you're still curious (but I warned you!), I recommend [9] as a general introduction to mathematical logic and axiomatic set theory (covering MK as well) and [13] as a discussion of set-theoretic foundations for category theory. In practice, most people basically use naive set theory but (try to) avoid typical traps like Russell's paradox by accepting that there is this notion of classes, that you can work with classes like with sets, that every set is a class, and that you can form the class of all sets. This is what we will do here as well. Then the correct way to state Definition 1.1.1 is to replace "collection" by "class" and everything is precise and works. Deal?

1.5. Smallness

It is useful to introduce some terminology to describe categories which are not "too big" and fit into the lower hierarchy of sets. A category $\mathcal{C}$ is called **small** if both the object class $\mathrm{Ob}_{\mathcal{C}}$ and the morphism class $\mathrm{Hom}_{\mathcal{C}}$ are actually sets. Most real-life examples—like all in Table 1.1—are not small, however, since the object class is a proper class. But what is true in all examples in Table 1.1 is that at least the class $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms between any two objects is a set, e.g. if you have two sets X and Y, then the class of maps between X and Y is the Cartesian product $\prod_{x \in X} Y$, which is a set. Such categories are called **locally small**. Depending on the literature, categories may be locally small by definition. Of course, subcategories of (locally) small categories are (locally) small as well.

Let's look at the category set of finite sets. It is of course locally small. But is it small? I have to disappoint you: the class of finite sets is again a proper class! Even the class of all one-element sets is a proper class! Here is how to see this. The

axiom of union in ZF states that for any set A there is a set $\bigcup A$ which consists of the elements of the elements of the set A. So, if the class A of one-element sets were a set, then $\bigcup A$ would be a set as well. But $\bigcup A$ contains all sets (think about it), so we would run into Russell's paradox!

But still there is some "smallness" to set compared to Set, namely when we consider objects up to isomorphism. Let's make this precise for a general category $\mathcal{C}$. Since we can work with classes like with sets, we can consider the relation on $\mathrm{Ob}_{\mathcal{C}}$ given by isomorphism between objects, and then we can form the class $[\mathcal{C}]$ of equivalence classes of this relation. These equivalence classes are called **isomorphism classes** of $\mathcal{C}$, and the isomorphism class of an object X will be denoted by $[X]$. Now, we say that the category $\mathcal{C}$ is **essentially small** if it is locally small and the class $[\mathcal{C}]$ is a set. Note that a *full* subcategory of an essentially small category is again essentially small; but a non-full subcategory does not need to be because when we have less morphisms, we can have more isomorphism classes.

Example 1.5.1. The category set of finite sets is essentially small: sets up to isomorphism are uniquely described by their cardinality, hence there is a bijection $[\mathsf{set}] \to \mathbb{N}$, so $[\mathsf{set}]$ is a set. On the other hand, the big brother Set is not essentially small: the class $[\mathsf{Set}]$ is (isomorphic to) the class of all cardinal numbers and one can show that this is not a set. Add reference

Example 1.5.2. The category $K\text{-}\mathsf{vec}$ of finite-dimensional K-vector spaces is essentially small: vector spaces up to isomorphism are described by their dimension, hence there is a bijection $[K\text{-}\mathsf{vec}] \to \mathbb{N}$, so $[K\text{-}\mathsf{vec}]$ is a set. Similarly as above, the big brother $K\text{-}\mathsf{Vec}$ is not essentially small.

Example 1.5.3. Generalizing Example 1.5.2, the category $R\text{-}\mathsf{mod}$ of finitely generated R-modules is essentially small. Since modules over a general ring do not need to have a basis, we cannot argue with the dimension as we did with vector spaces. Instead, we can prove this as follows. If M is generated by finitely many elements $m_1, \ldots, m_n$, then we have a surjective morphism $R^n \to M$ mapping the i-th standard basis vector e_i to m_i. Hence, M is isomorphic to a quotient of R^n. Quotients of R^n are determined by the submodule we're modding out. Submodules of R^n are special subsets, so they are all members of the power set $\mathcal{P}(R^n)$ of R^n. This is a set, so the quotients of R^n form a set. Since $\bigcup_{n \in \mathbb{N}} \mathcal{P}(R^n)$ is a set as well, the quotients of all R^n for $n \in \mathbb{N}$ form a set. The isomorphism classes of finitely generated R-modules are uniquely described by members of this set, so $[R\text{-}\mathsf{mod}]$ is a set. As before, the big brother $[R\text{-}\mathsf{Mod}]$ is not essentially small (except for if R is the zero ring).

CHAPTER 2

Functors

In the introduction of Chapter 1 I said that whenever you have a mathematical structure you also want to consider morphisms between them. This observation led us to the notion of categories. But now a category itself is also a mathematical structure, so is there a notion of "morphism" between them? Yes, of course! You would consider a morphism of the underlying quivers (so, you map objects to objects and morphisms to morphisms while respecting source and target) and this has to respect the composition and the identity as well. Instead of "morphism" between categories one uses the fancier term "functor".

2.1. Definition and basic examples

Let's formalize this idea.

Definition 2.1.1. A **functor** $F\colon \mathcal{C} \to \mathcal{C}'$ between categories $\mathcal{C}$ and $\mathcal{C}'$ consists of:

- a map $F_0\colon \mathcal{C}_0 \to \mathcal{C}'_0$, simply denoted $X \mapsto F(X)$,
- a map $F_1\colon \mathcal{C}_1 \to \mathcal{C}'_1$, simply denoted $f \mapsto F(f)$,

such that the following holds:

(1) F is a morphism $Q(\mathcal{C}) \to Q(\mathcal{C}')$ of the underlying quivers, i.e.

$$F_0 \circ s = s' \circ F_1 \quad \text{and} \quad F_0 \circ t = t' \circ F_1 \;, \tag{2.1}$$

(2) F is compatible with the composition, i.e.

$$F(g \circ f) = F(g) \circ F(f) \tag{2.2}$$

for any composable pair $f, g \in \mathcal{C}_1$,

(3) F preserves the identity, i.e.

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \tag{2.3}$$

for any $X \in \mathcal{C}_0$.

A functor F induces "local" maps

$$F_{X,Y}\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X),F(Y)) \tag{2.4}$$

for any pair of objects. As with categories it is in practice often more convenient to specify the action of a functor on morphisms by such maps.

Example 2.1.2. Every group G has an underlying set $F(G)$ and every group morphism $f\colon G \to H$ has an underlying map $F(f)\colon F(G) \to F(H)$ of sets. It is clear that this process is compatible with composition and preserves the identity. We thus have a functor $F\colon \mathsf{Grp} \to \mathsf{Set}$. This functor simply "forgets" about the group structure and is therefore called a **forget functor**. For many algebraic structures with an underlying set—like all in Table 1.1—you have an obvious forget functor to Set. You can also forget other structure, e.g. you can forget the scalar action on a vector space and get a forget functor $K\text{-}\mathsf{Vec} \to \mathsf{Ab}$.

EXAMPLE 2.1.3. Examples like the forget functor $K\text{-}\mathsf{Vec} \to \mathsf{Set}$ above are maybe a bit silly. But we can also construct a less silly one in the opposite direction. Take a set X. We can then consider the K-vector space

$$K^{(X)} := \bigoplus_{x \in X} K := \left\{ (\alpha_x)_{x \in X} \in \prod_{x \in X} K \mid \text{all but finitely many } \alpha_x = 0 \right\} . \tag{2.5}$$

This is really just the K-vector space with a basis $\{e_x\}_{x \in X}$ indexed by the elements of X. This gives us a map $\mathsf{Set} \to K\text{-}\mathsf{Vec}$ on objects. To make this into a functor, we also need to define a mapping on morphisms. So, let $\varphi \colon X \to Y$ be a map of sets. We can define a linear map $f \colon K^{(X)} \to K^{(Y)}$ sending e_x to $e_{\varphi(x)}$. It is easy to see that the mapping $\varphi \mapsto f$ is compatible with composition and respects the identity. We have thus defined a functor $\mathsf{Set} \to K\text{-}\mathsf{Vec}$. In a similar way you can more generally define a functor $\mathsf{Set} \to R\text{-}\mathsf{Mod}$.

EXERCISE 2.1.4. Define a functor $\mathsf{Set} \to \mathsf{Grp}$ analogous to Example 2.1.3.

EXAMPLE 2.1.5. Fix $n \in \mathbb{N}$. We want to show that taking the general linear group GL_n over a ring yields a functor $\mathrm{GL}_n \colon \mathsf{Ring} \to \mathsf{Grp}$. Recall that an $n \times n$-matrix A over R is said to be **invertible** if its determinant $\det(A)$ is a unit in R. Since the determinant is multiplicative, the set $\mathrm{GL}_n(R)$ of invertible $n \times n$-matrices over R forms a group under matrix multiplication, called the **general linear group**. We thus have a map $\mathrm{GL}_n \colon \mathsf{Ring} \to \mathsf{Grp}$ between objects. Always remember that to specify a functor we also need to specify an action on morphisms. So, to a ring morphism $f \colon R \to S$ we need to associate a group morphism $\mathrm{GL}_n(f) \colon \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$. There's an obvious thing to do. Applying f to the entries of $A \in \mathrm{GL}_n(R)$ yields an $n \times n$-matrix $f(A)$ over S. This matrix is invertible as well since $\det(f(A)) = f(\det(A))$ and a ring morphism maps units to units. The resulting map

$$\begin{array}{rcl} \mathrm{GL}_n(f) \colon \mathrm{GL}_n(R) & \to & \mathrm{GL}_n(S) \\ A & \mapsto & f(A) \end{array}$$

is easily seen to be a group morphism. It is evident that the mapping $f \mapsto \mathrm{GL}_n(f)$ is compatible with composition and preserves the identity, so we have constructed a functor $\mathrm{GL}_n \colon \mathsf{Ring} \to \mathsf{Grp}$.

EXAMPLE 2.1.6. Let $\mathcal{C}$ be a locally small category. Then for every object $X \in \mathcal{C}$ we have the **Hom-functor**

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathsf{Set} \tag{2.6}$$

which maps $Y \in \mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and which maps a morphism $f \colon Y \to Z$ in $\mathcal{C}$ to the set map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(X, f) \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) & \to & \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ g & \mapsto & f \circ g \, . \end{array} \tag{2.7}$$

A fundamental property of a functor $F \colon \mathcal{C} \to \mathcal{C}'$ is that it maps isomorphisms to isomorphisms. Namely, if $f \colon X \to Y$ is an isomorphism in $\mathcal{C}$ with inverse g, then

$$\mathrm{id}_{F(X)} = F(\mathrm{id}_X) = F(g \circ f) = F(g) \circ F(f) \tag{2.8}$$

and

$$\mathrm{id}_{F(Y)} = F(\mathrm{id}_Y) = F(f \circ g) = F(f) \circ F(g) \ , \tag{2.9}$$

hence $F(f)\colon F(X) \to F(Y)$ is an isomorphism with inverse $F(g)$. Hence, F induces a map

$$[F]\colon [\mathcal{C}] \to [\mathcal{C}'] \tag{2.10}$$

between the classes of isomorphism classes of objects, so a functor can in particular be viewed as giving a $\mathcal{C}'$-valued **invariant** for objects of $\mathcal{C}$. For example, the general linear group functor from Example 2.1.5 gives a group-valued invariant of rings: isomorphic rings have isomorphic general linear groups. This particular example is probably not so spectacular and surprising but if you have a more complicated or interesting functor, this can be quite dramatic. Algebraic topology is basically the construction and study of functors $\mathsf{Top} \to \mathsf{Grp}$, which give you an algebraic handle to study topological spaces.

EXERCISE 2.1.7. Some functors F have the property that they **reflect** isomorphisms, i.e. if $F(f)$ is an isomorphism, then so is f. Find examples of such functors. Show that this is *not* a general feature of functors, however.

2.2. The co-world

In practice, one often encounters maps $F_0\colon \mathcal{C}_0 \to \mathcal{C}'_0$ and $F_1\colon \mathcal{C}_1 \to \mathcal{C}'_1$ between objects and morphisms of categories which basically define a functor except for one difference: they *reverse* source and target. So, (2.1) becomes

$$F_0 \circ s = t' \circ F_1 \quad \text{and} \quad F_0 \circ t = s' \circ F_1 \ , \tag{2.11}$$

we thus have maps

$$F_{X,Y}\colon \ \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(Y),F(X)) \ , \tag{2.12}$$

and the compatibility with composition becomes

$$F(g \circ f) = F(f) \circ F(g) \ . \tag{2.13}$$

Such a thing is called a **contravariant functor** in contrast to a usual functor as defined above, which is then sometimes for emphasis called a **covariant functor**.

EXAMPLE 2.2.1. Let V be a K-vector space. The **dual** of V is the K-vector space $V^* := \mathrm{Hom}_K(V,K)$ of K-linear maps $V \to K$. This is a K-vector space with respect to pointwise addition and scalar multiplication of maps. A linear map $f\colon V \to W$ induces a linear map

$$\begin{aligned} f^*\colon W^* \quad &\to \quad V^* \\ \varphi \quad &\mapsto \quad \varphi \circ f \ . \end{aligned} \tag{2.14}$$

So, we turn a function on W to a function on V by first applying the map φ to get from V to W. Such a construction is called a **pullback** of functions and this naturally reverses directions as you can see. The maps $V \mapsto V^*$ and $f \mapsto f^*$ define a *contravariant* functor $(-)^*\colon K\text{-}\mathsf{Vec} \to K\text{-}\mathsf{Vec}$.

EXAMPLE 2.2.2. In a way similar as above, consider a topological space X and let $C(X)$ be the ring of real valued continuous functions $X \to \mathbb{R}$. This is a ring with respect to pointwise addition and multiplication. We thus have a map $C\colon\ \mathsf{Top} \to \mathsf{Ring}$ on objects. If $f\colon X \to Y$ is a continuous map of topological spaces, then you can check that pullback of functions from Y to X, i.e. the map $\varphi \mapsto \varphi \circ f$, defines a ring morphism $C(f)\colon C(Y) \to C(X)$. In total, we get a *contravariant* functor $C\colon \mathsf{Top} \to \mathsf{Ring}$.

Example 2.2.3. Recall the Hom-functor from Example 2.1.6. For an object Y in a locally small category $\mathcal{C}$ we can also consider the **contravariant Hom-functor**

$$\mathrm{Hom}_{\mathcal{C}}(-,Y)\colon \mathcal{C} \to \mathsf{Set}\ , \tag{2.15}$$

which maps an object $X \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and which maps a morphism $f\colon X \to Z$ in $\mathcal{C}$ to the set map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f,Y)\colon\ \mathrm{Hom}_{\mathcal{C}}(Z,Y) &\ \to\ \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ g &\ \mapsto\ g \circ f\ . \end{aligned} \tag{2.16}$$

It would be annoying if we would have to consider covariant and contravariant functors separately. There's a *formal* trick that shows that any general fact about covariant functors holds similarly for contravariant functors as well. Namely, given a category $\mathcal{C}$ we formally define a new category $\mathcal{C}^{\mathrm{op}}$, called the **opposite category** of $\mathcal{C}$, as follows: the objects, morphisms, and composition are the same as for $\mathcal{C}$ but we exchange source and target functions, i.e.

$$s^{\mathrm{op}} := t \quad \text{and} \quad t^{\mathrm{op}} := s\ . \tag{2.17}$$

We thus have

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,X) = \mathrm{Hom}_{\mathcal{C}}(X,Y)\ . \tag{2.18}$$

Note that the above relation is only formal—we do not invert any morphisms. You can easily convince yourself that $\mathcal{C}^{\mathrm{op}}$ is a category and that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. The important (but obvious) upshot is now that a *contravariant* functor $\mathcal{C} \to \mathcal{C}'$ is precisely the same as a *covariant* functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}'$ or $\mathcal{C} \to \mathcal{C}'^{\mathrm{op}}$. In general discussions about functors, we will thus restrict to covariant functors, and then you can find the analogous concepts and results for contravariant functors easily yourself.

The opposite category exposes a fundamental property in category theory, namely that many concepts have an opposite existence. For example, when we discussed monomorphisms and epimorphisms in Section 1.3, did you notice that this is basically the same concept—just opposite? A monomorphism in $\mathcal{C}$ is an epimorphism when considered in $\mathcal{C}^{\mathrm{op}}$ and vice versa. In the terminology of concepts, the opposite of a certain concept often carries the prefix **co**. For example, a morphism has a domain and a codomain (the codomain is the domain in the opposite category). We could call an epimorphism a co-monomorphism or call a monomorphism a co-epimorphism but it is more convenient to use the two separate terms. A contravariant functor is sometimes called a co-functor. You know what a kernel of a morphism of vector spaces is and maybe you have heard of cokernels as well—more about this later. I think you got the idea.

2.3. The category of categories

We said that categories are also mathematical structures and this led us to think about morphisms between categories (the functors). But you already know that when you have a mathematical structure and a notion of morphisms between them, you usually get a category. So, is there a category of categories? Of course!

What we need to do is to define a composition of functors—and this is obvious: if $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}''$ are two functors, their **composition** is the functor

$$G \circ F\colon \mathcal{C} \to \mathcal{C}'' \tag{2.19}$$

with

$$(G \circ F)_0 = G_0 \circ F_0 \quad \text{and} \quad (G \circ F)_1 = G_1 \circ F_1 \ , \tag{2.20}$$

i.e. the maps on objects and on morphisms are simply the composition of those of F and G. And now I leave it up to you to check that the composition satisfies all the properties you need for a category, so you get a category Cat of categories. As it will show up many times, I note that the identity for the composition is the **identity functor**

$$\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C} \tag{2.21}$$

with $\mathrm{id}_{\mathcal{C}}(X) = X$ for all $X \in \mathcal{C}$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}$.

If you are careful, you may have noticed that in the definition of Cat we ran again into set-theoretic issues as discussed in Section 1.4. Namely, given any class $\mathcal{C}$, we can upgrade it to a category simply by formally adding identity morphisms and taking the trivial composition. Hence, the collection of all categories contains all classes—and now you know that this is not a class but a bigger thing we called conglomerate. Since we agreed that the correct way to state the definition of a category is to replace "collection" by "class", our Cat is unfortunately not a category in this sense—it's too big. The usual way to deal with this issue is to consider only *small* categories instead of *all* categories. The collection of small categories forms a class (think about it), so you get a well-defined honest category of small categories. I call this approach **size restriction**—and this is what we'll do here.

Somehow this seems unsatisfactory because we have a working concept of general (non-small) categories and functors between them. Remember that sets, classes, and conglomerates are just different levels in a type hierarchy. Let's assign them numbers instead of fancy names: size-0 stands for sets, size-1 for classes, size-2 for conglomerates, and now you can continue this. We can base our notion of "collection" in the formal definition of a category on any of these sizes. We then have a size-1 category Cat_0 of size-0 categories (this is the category of small categories), a size-2 category Cat_1 of size-1 categories, etc. We can now consider a size-$(n+1)$ category Quiv_n of size-n quivers and get a functor $\mathsf{Cat}_n \to \mathsf{Quiv}_n$ associating to a category its underlying quiver (think about it). So, overall, size restriction is not really restrictive when you carefully attach and consider sizes everywhere. In practice, no one does this because it's just annoying.

The category of categories point of view is very helpful. For example, you immediately know what an **isomorphism** of categories should be: it is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ such that there is a functor $G\colon \mathcal{C}' \to \mathcal{C}$ with $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{C}'}$.

EXAMPLE 2.3.1. Remember the category $\mathsf{Rep}_K(G)$ of K-linear representations of a group G. We will show that this category is naturally isomorphic to a category of modules over a certain ring. Namely, let KG be the K-vector space with a basis $(\delta_g)_{g \in G}$ indexed by the elements of G. We turn this into a ring by defining

$$\delta_g \cdot \delta_h \coloneqq \delta_{gh} \ , \tag{2.22}$$

and then extend linearly to general elements $\sum_g a_g \delta_g$ of KG. This is the **group ring** of G over K. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. We define a KG-module structure on V via $\delta_g v := \rho(g)(v)$. If $f\colon V \to V'$ is a morphism of representations, then it is also a KG-module morphism. This defines a functor $\mathsf{Rep}_K(G) \to KG\text{-}\mathsf{Mod}$. Conversely, a KG-module V defines a representation $\rho\colon G \to \mathrm{GL}(V)$ via $\rho(g)(v) := \delta_g v$. If $V \to V'$ is a KG-module morphism, it is also a morphism of the associated representations. This defines a functor $KG\text{-}\mathsf{Mod} \to \mathsf{Rep}_K(G)$. It is evident that the two functors are inverse to each other, so the categories $\mathsf{Rep}_K(G)$ and $KG\text{-}\mathsf{Mod}$ are isomorphic. This is a very helpful fact since now we can apply general results for modules over rings to representations. Because the isomorphism is so natural and obvious, we won't even mention it explicitly from now on any more: representations and KG-modules are the same thing.

2.4. Equivalence of categories and morphisms of functors

You may be surprised when I tell you that the notion of isomorphism of categories is actually not the right thing to look at—it is too strong and rarely occurs. The problem is the equal sign in $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{C}'}$. In this course you will slowly start to learn that when passing from set-theoretic concepts to categorical concepts, equal signs should better become something up to isomorphism.[1] Let's look at an example that makes this clear.

Example 2.4.1. Let's consider the category $K\text{-}\mathsf{vec}$ of finite-dimensional vector spaces over K. You know that after choosing bases, any vector space is isomorphic to some K^n and any linear map can be encoded by a matrix with respect to the chosen bases. Lets define a category $K\text{-}\mathsf{mat}$ as follows: the objects are the natural numbers $\mathbb{N}$, the morphisms $n \to m$ are the $(m \times n)$-matrices over K, and the composition is matrix multiplication. Basically, the category $K\text{-}\mathsf{mat}$ is the same as the category $K\text{-}\mathsf{vec}$—but only up to isomorphism of objects. Let's look at this more closely. We surely have a functor

$$F\colon K\text{-}\mathsf{mat} \to K\text{-}\mathsf{vec} \tag{2.23}$$

mapping n to K^n and mapping a morphism $n \to m$, i.e. an $(m \times n)$-matrix, to the corresponding linear map $K^n \to K^m$ in the standard basis. But this functor is not surjective since we only reach the vector spaces K^n and not all its isomorphic friends. Nonetheless, we can construct a functor

$$G\colon K\text{-}\mathsf{vec} \to K\text{-}\mathsf{mat} \tag{2.24}$$

in the opposite direction as follows:

(1) we map $V \in K\text{-}\mathsf{vec}$ to $\dim_K(V)$;
(2) using the axiom of choice on the class $K\text{-}\mathsf{vec}$ we choose for each $V \in K\text{-}\mathsf{vec}$ a basis and then map a morphism $f\colon V \to W$ to the matrix in the chosen basis.

We have $FG(V) = K^n$ with $n = \dim_K(V)$. This is not necessarily equal to V, i.e. $FG \neq \mathrm{id}$, so G is not an inverse to F (it can't be of course). But from our choice

[1] I highly recommend the Quanta Magazine article "With Category Theory, Mathematics Escapes From Equality" at `https://www.quantamagazine.org/with-category-theory-mathematics-escapes-from-equality-20191010/`.

of bases we at least get an isomorphism

$$\varepsilon_V : FG(V) = K^n \overset{\simeq}{\to} V = \mathrm{id}(V)\ . \tag{2.25}$$

These isomorphisms have the following naturality property: if $f\colon V \to W$ is a morphism in K-vec, then the diagram

$$\begin{array}{ccc} FG(V) & \xrightarrow{\varepsilon_V} & \mathrm{id}(V) \\ {\scriptstyle FG(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}(f)} \\ FG(W) & \xrightarrow[\varepsilon_W]{} & \mathrm{id}(W) \end{array} \tag{2.26}$$

commutes. Hence, even though FG is not *equal* to the identity functor, there is a way to *naturally transform* FG to the identity functor.

This should be motivation enough to introduce the following concepts.

Definition 2.4.2. Let $F, F'\colon \mathcal{C} \to \mathcal{C}'$ be two functors. A **natural transformation** (or **morphism**) from F to F' is a family

$$\eta := \{\eta_X : F(X) \to F'(X)\}_{X\in\mathcal{C}} \tag{2.27}$$

of morphisms in $\mathcal{C}'$ which is **natural**, i.e. for each morphism $f\colon X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccccc} X & \quad & F(X) & \xrightarrow{\eta_X} & F'(X) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle F'(f)} \\ Y & & F(Y) & \xrightarrow[\eta_Y]{} & F'(Y) \end{array} \tag{2.28}$$

commutes.

There's an obvious way to compose two morphisms $\eta\colon F \to F'$ and $\eta'\colon F' \to F''$ to a morphism $\eta' \circ \eta\colon F \to F''$, namely via

$$(\eta' \circ \eta)_X := \eta'_X \circ \eta_X : F(X) \to F''(X) \tag{2.29}$$

Moreover, on each functor $F\colon \mathcal{C} \to \mathcal{C}'$ we have an identity $\mathrm{id}_F\colon F \to F$ defined by

$$(\mathrm{id}_F)_X = \mathrm{id}_{F(X)}\ . \tag{2.30}$$

We can thus define a category $\mathsf{Fun}(\mathcal{C}, \mathcal{C}')$, called **functor category**, whose objects are the functors $\mathcal{C} \to \mathcal{C}'$, the morphisms are the morphisms of functors, and the composition is the composition just defined. From this point of view we immediately get a notion of **isomorphism** of functors $F, F'\colon \mathcal{C} \to \mathcal{C}'$, namely this is a morphism $\eta\colon F \to F'$ such that there is a morphism $\varepsilon\colon F' \to F$ with

$$\varepsilon \circ \eta = \mathrm{id}_F \quad \text{and} \quad \eta \circ \varepsilon = \mathrm{id}_{F'}\ . \tag{2.31}$$

Exercise 2.4.3. Show that a morphism $\eta\colon F \to F'$ of functors $\mathcal{C} \to \mathcal{C}'$ is an isomorphism if and only if η_X is an isomorphism for all $X \in \mathcal{C}$.

Now, we come to the key definition which is motivated by Example 2.4.1.

Definition 2.4.4. An **equivalence** of categories $\mathcal{C}$ and $\mathcal{C}'$ consists of functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ and isomorphisms $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{C}'}$ and $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$.

People are lazy and usually don't specify all the data of an equivalence. We say that $\mathcal{C}$ and $\mathcal{C}'$ are **equivalent** if there is an equivalence between them. We say that $F\colon \mathcal{C} \to \mathcal{C}'$ is an **equivalence** if it can be upgraded to an equivalence. A functor $G\colon \mathcal{C}' \to \mathcal{C}$ such that the pair (F, G) can be upgraded to an equivalence is called a **weak inverse** of F.

EXAMPLE 2.4.5. An isomorphism $F\colon \mathcal{C} \to \mathcal{C}'$ of categories is obviously also an equivalence—more precisely, the datum $(F, F^{-1}, \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}, \mathrm{id}_{\mathrm{id}_{\mathcal{C}'}})$ is an equivalence.

EXAMPLE 2.4.6. The datum (F, G, ε) constructed in Example 2.4.1 can be upgraded to an equivalence between K-mat and K-vec. The only missing piece of information is an isomorphism $\eta\colon \mathrm{id} \to GF$. We have $GF(n) = n$ already, i.e. GF is the identity on objects. A morphism $n \to m$ in K-Mat is a matrix $M \in \mathrm{Mat}_{m\times n}(K)$. Associated to this is the linear map $F(M)\colon K^n \to K^m$ in the standard bases. But now $GF(M)$ is not necessarily equal to M because in our choice of bases on vector spaces we did not require to choose the standard basis on K^n. Of course we can require this particular choice and get an appropriate G with $GF(M) = M$. Then we have $\mathrm{id} = GF$ also on morphisms, so $\eta\colon \mathrm{id} \to GF$ will be the identity morphism.

Note that in contrast to an actual inverse, a *weak* inverse of an equivalence is not unique, e.g. in Example 2.4.6 you get a weak inverse for any choice of bases. But:

LEMMA 2.4.7. *A weak inverse of an equivalence is unique up to isomorphism.*

PROOF. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an equivalence and let $G, G'\colon \mathcal{C}' \to \mathcal{C}$ be weak inverses. Then we have in particular isomorphisms $\varepsilon'\colon FG' \to \mathrm{id}$ and $\eta\colon \mathrm{id} \to GF$. For $X \in \mathcal{C}$ we get an isomorphism $\varepsilon'_X\colon FG'(X) \to X$. Since functors preserve isomorphisms, applying G yields an isomorphism $G(\varepsilon'_X)\colon GFG'(X) \to G(X)$. On the other hand, we have an isomorphism $\eta_{G'(X)}\colon G'(X) \to GFG'(X)$. The composition of the two isomorphisms gives an isomorphism $G'(X) \to G(X)$. I leave it to you to check that this family of isomorphisms is natural and thus an isomorphism $G' \to G$. □

There's a more convenient way to prove that a functor is an equivalence which is apparent also in Example 2.4.1.

DEFINITION 2.4.8. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called:

- **faithful** if $F_{X,Y}$ is injective for all X, Y;
- **full** if $F_{X,Y}$ is surjective for all X, Y;
- **fully faithful** if it is full and faithful, i.e. $F_{X,Y}$ is bijective for all X, Y;
- **essentially surjective** if any $X' \in \mathcal{C}'$ is isomorphic to $F(X)$ for some $X \in \mathcal{C}$.

EXAMPLE 2.4.9. The functor $F\colon K\text{-mat} \to K\text{-vec}$ from Example 2.4.1 is fully faithful and essentially surjective.

EXERCISE 2.4.10. Show that a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence if and only if it is fully faithful and essentially surjective.

If in the definition of an equivalence we consider contravariant functors instead, we speak—depending on the author—of a **contravariant equivalence**, or **anti-equivalence** or **duality**.

EXAMPLE 2.4.11. Recall the contravariant functor

$$(-)^* := \mathrm{Hom}_K(-, K)\colon K\text{-}\mathsf{vec} \to K\text{-}\mathsf{vec} \tag{2.32}$$

from Example 2.2.1. This is a duality with a weak inverse being the functor itself. Namely, for any vector space V there is a canonical isomorphism

$$\varepsilon_V : V \to V^{**} \tag{2.33}$$

given by mapping $v \in V$ to the linear map $\varepsilon_V(v) \in V^{**} = \mathrm{Hom}_K(V^*, K)$ which maps $f \in V^*$ to $f(v)$. This yields an isomorphism $\mathrm{id} \to (-)^* \circ (-)^*$. You can take its inverse $(-)^* \circ (-)^* \to \mathrm{id}$ to get all the data of a duality.

Note that if $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$, then we have a natural inclusion functor $\mathcal{C} \to \mathcal{C}'$, and this is fully faithful. Conversely, from the characterization of an equivalence in Exercise 2.4.10, it is clear that a fully faithful functor $F\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence between $\mathcal{C}$ and its **full image**, which is the full subcategory of $\mathcal{C}'$ formed by the objects $F(X)$ for $X \in \mathcal{C}$. A fully faithful functor is therefore also called an **embedding**.

EXAMPLE 2.4.12. Let $\mathcal{C}$ be a locally small category. Recall from Example 2.1.6 that for every $X \in \mathcal{C}$ we have the Hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathsf{Set}$. Convince yourself that mapping X to $\mathrm{Hom}_{\mathcal{C}}(X, -)$ can be made into a functor

$$h^- : \mathcal{C} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Set}) \ . \tag{2.34}$$

The **Yoneda lemma** states that this functor is an embedding. I'll leave it up to you to prove this—it's really not that difficult. We can thus identify $\mathcal{C}$ with a full subcategory of $\mathsf{Fun}(\mathcal{C}, \mathsf{Set})$. The functors in the image are also called **representable** functors because they are "represented" by an object of X.

Give more details and a proof.

There is also a contravariant version where you consider the covariant (!) functor

$$h_- : \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \tag{2.35}$$

that you get by mapping $Y \in \mathcal{C}$ to the contravariant (!) functor $\mathrm{Hom}_{\mathcal{C}}(-, Y)$.

EXERCISE 2.4.13. Prove the Yoneda lemma.

REMARK 2.4.14. It depends on the author what exactly is meant by "embedding": for some it's a fully faithful functor (like for us), for some it's a faithful functor that is injective on objects, and for some it's a fully faithful functor that is injective on objects. I find our definition best because it fits what you get from important embedding theorems like the Yoneda lemma or the Mitchell embedding theorem.

REMARK 2.4.15. You may first want to define the image of a functor $F\colon \mathcal{C} \to \mathcal{C}'$ as the subcollection consisting of all objects $F(X)$ for $X \in \mathcal{C}$ and of all morphisms $F(f)$ for $f \in \mathrm{Hom}_{\mathcal{C}}$. But this is in general not a subcategory because the collection of morphisms you get is not necessarily closed under composition. One thus defines the **image** of F to be the smallest subcategory of $\mathcal{C}'$ containing the $F(X)$ and $F(f)$. The **full image** is defined as the full subcategory containing all the $F(X)$. If F is full, then the image is equal to the full image.

EXAMPLE 2.4.16. A category $\mathcal{C}$ is called **skeletal** if each object has a singleton isomorphism class, i.e. for each $X \in \mathcal{C}$ the only object isomorphic to X is X itself. A **skeleton** of a category $\mathcal{C}'$ is a skeletal full subcategory $\mathcal{C}$ of $\mathcal{C}'$. By construction, the inclusion functor $\mathcal{C} \to \mathcal{C}'$ is fully faithful and essentially surjective, hence it is an equivalence, i.e. $\mathcal{C}'$ is equivalent to any skeleton. A skeleton always exists: you can get one by choosing (using the axiom of choice) a fixed representative from each isomorphism class of objects. A skeleton of Set is the full subcategory of cardinal numbers. Similarly, a skeleton of K-Vec is the full subcategory formed by the K^α for α a cardinal number.

EXERCISE 2.4.17. Show that two categories are equivalent if and only if they have isomorphic skeleta.

2.5. Adjoint functors

There's a weaker concept of equivalence of categories which arises basically everywhere in mathematics: adjunctions. This concept was introduced by Daniel Kan in 1958. Again, we start with an example.

EXAMPLE 2.5.1. Consider the forget functor K-$\mathsf{Vec} \to \mathsf{Set}$ and let's denote it by G this time (think of "forGet"). This functor is not an equivalence since, e.g., it is not full (clearly); also, it is not essentially surjective (e.g. $\emptyset$ is not in the essential image, or if $K = \mathbb{Q}$ then no finite set with more than two elements is in the essential image, etc.). Nonetheless we have constructed in Example 2.1.3 a functor $F\colon \mathsf{Set} \to K$-$\mathsf{Vec}$ in the opposite direction mapping a set X to the vector space $K^{(X)}$ with basis indexed by X and mapping a map $\varphi\colon X \to Y$ of sets to the linear map $f\colon K^{(X)} \to K^{(Y)}$ which maps the standard basis vector e_x to $e_{\varphi(x)}$.

Of course, F cannot be a weak inverse of G as there is none. But still, there's a nice relation between F and G, and this is as follows: because $K^{(X)}$ has basis $\{e_x\}_{x\in X}$ it follows that giving a linear map $K^{(X)} \to V$ out of $K^{(X)}$ amounts precisely to giving a map $X \to G(V)$ of sets. Namely, any map $\varphi\colon X \to G(V)$ induces a linear map $f\colon K^{(X)} \to V$ by mapping e_x to $e_{\varphi(x)}$, and conversely if $f\colon K^{(X)} \to V$ is a linear map we get a map $\varphi\colon X \to G(V)$ sending x to $f(e_x)$. Hence, for any set X and any vector space V we have a canonical bijection

$$\begin{array}{ccc} \mathrm{Hom}_{K\text{-}\mathsf{Vec}}(F(X), V) & \simeq & \mathrm{Hom}_{\mathsf{Set}}(X, G(V)) \\ f & \leftrightarrow & \varphi \end{array} \tag{2.36}$$

These bijections are natural in the arguments X and V, i.e. if $\varphi\colon X \to Y$ is a map of sets and $f\colon V \to W$ is a linear map, then the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{K\text{-}\mathsf{Vec}}(F(Y), V) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathsf{Set}}(Y, G(V)) \\ {\scriptstyle g\mapsto f\circ g\circ F(\varphi)}\downarrow & & \downarrow{\scriptstyle \psi\mapsto G(f)\circ\psi\circ\varphi} \\ \mathrm{Hom}_{K\text{-}\mathsf{Vec}}(F(X), W) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathsf{Set}}(X, G(W)) \end{array} \tag{2.37}$$

commutes.

There's a more efficient way to state (2.36) and (2.37).

DEFINITION 2.5.2. Given categories $\mathcal{C}_1$ and $\mathcal{C}_2$, we define their **product** $\mathcal{C}_1 \times \mathcal{C}_2$ to be the category with:

- objects being pairs (X_1, X_2) of objects $X_i \in \mathcal{C}_i$;
- morphisms from (X_1, X_2) to (X_1', X_2') being pairs (f_1, f_2) of morphisms $f_i \in \mathrm{Hom}_{\mathcal{C}_i}(X_i, X_i')$;
- composition being component-wise, i.e. $(f_1', f_2') \circ (f_1, f_2) = (f_1' \circ f_1, f_2' \circ f_1)$;
- identity being $\mathrm{id}_{(X_1, X_2)} = (\mathrm{id}_{X_1}, \mathrm{id}_{X_2})$.

A functor $F\colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ is also called a **bifunctor** from $\mathcal{C}_1$ and $\mathcal{C}_2$ to $\mathcal{D}$.

Now, given a category $\mathcal{C}$, you can easily convince yourself that

$$\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set} \tag{2.38}$$

is a functor and that we can restate properties (2.36) and (2.37) by saying that there is an isomorphism of functors

$$\mathrm{Hom}_{K\text{-}\mathsf{Vec}}(F(-),-) \simeq \mathrm{Hom}_{\mathsf{Set}}(-,G(-)) \ . \tag{2.39}$$

This should be motivation enough for the following definition.

Definition 2.5.3. An **adjunction** between categories $\mathcal{C}$ and $\mathcal{C}'$ consists of a pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}' \tag{2.40}$$

together with an isomorphism

$$\Phi\colon \mathrm{Hom}_{\mathcal{C}'}(F(-),-) \overset{\simeq}{\Rightarrow} \mathrm{Hom}_{\mathcal{C}}(-,G(-)) \ . \tag{2.41}$$

A pair (F,G) of functors is called an **adjoint pair** if it can be upgraded to an adjunction. A **right adjoint** to a functor F is a functor G such that (F,G) is an adjoint pair; and a **left adjoint** to a functor G is a functor F such that (F,G) is an adjoint pair.

The bijection (2.41) really just says that for objects $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$ the morphisms $F(X) \to X'$ are in bijection with morphisms $X \to G(X')$, and these bijections are natural. This can become very effective in practice when you have good knowledge about morphisms in one of the categories.

Example 2.5.4. The functor $F\colon \mathsf{Set} \to K\text{-}\mathsf{Vec}$ from Example 2.5.1 is a left adjoint to the forget functor $G\colon K\text{-}\mathsf{Vec} \to \mathsf{Set}$.

There's a more general concept behind Example 2.5.4. Consider a category $\mathcal{C}$ with a faithful functor $G\colon \mathcal{C} \to \mathsf{Set}$. Such categories are called **concrete** because you can think of G as giving an underlying set, resp. set map, for the objects of $\mathcal{C}$, resp. the morphisms in $\mathcal{C}$. All the examples in Table 1.1 are concrete. If G has a left adjoint $F\colon \mathsf{Set} \to \mathcal{C}$ this means that for any set X we can construct an object $F(X)$ such that morphisms $F(X) \to V$ in $\mathcal{C}$ are in bijection with set maps $X \to G(V)$, i.e. morphisms out of $F(X)$ are completely described by set maps out of X. One therefore says that $F(X)$ is a **free object** on the set X. Such a construction exists in many examples, see Table 2.1.

Exercise 2.5.5. Spell out the left adjoint to the forget functor to Set in the examples in Table 2.1 and prove that it is indeed a left adjoint.

Exercise 2.5.6. Find a left adjoint F to the functor $G\colon \mathsf{Cat} \to \mathsf{Quiv}$ associating to a category its underlying quiver. For a quiver Q one calls $F(Q)$ the **free category** on Q.

The free constructions (left adjoint to a forget functor) are all nice and stuff but are there any other examples of adjunctions? Yes, and if you start looking for them you'll get flooded with adjunctions! I'll just give a few more examples here.

Category	Free object
Mon	free monoid X^*
Grp	free group $\langle X\rangle$
Ab	free abelian group $\mathbb{Z}^{(X)}$
K-Vec	vector space $K^{(X)}$
Ring	tensor algebra $\mathbb{Z}\langle X\rangle$
CRing	polynomial ring $\mathbb{Z}[X]$
R-Mod	free module $R^{(X)}$
R-Alg	tensor algebra $R\langle X\rangle$
Top	discrete topology on X

TABLE 2.1. Free objects (giving a left adjoint to the forget functor).

EXAMPLE 2.5.7. We have an inclusion functor $i\colon \mathsf{Ab}\to\mathsf{Grp}$ from the category of abelian groups into the category of groups. This has a left adjoint $\mathsf{Grp}\to\mathsf{Ab}$, namely **abelianization**. To a group G you map the abelian group

$$G^{\mathrm{ab}} \coloneqq G/[G,G]\ , \tag{2.42}$$

where $[G,G]$ is the subgroup of G generated by commutators, i.e. elements of the form $[g,h]\coloneqq ghg^{-1}h^{-1}$. For obvious reasons, $[G,G]$ is called the **commutator subgroup** of G. If $f\colon G\to G'$ is a group morphism, then $f([G,G])\subseteq[G',G']$, so f induces a morphism $G^{\mathrm{ab}}\to(G')^{\mathrm{ab}}$. This defines a functor $(-)^{\mathrm{ab}}\colon \mathsf{Grp}\to\mathsf{Ab}$.

Now, if G is a group and A is an *abelian* group, then a group morphism $f\colon G\to A$ always has the commutator subgroup of G in its kernel, hence it induces a morphism $G^{\mathrm{ab}}\to A$, and any such morphism arises in this way. This correspondence is natural, hence

$$\mathrm{Hom}_{\mathsf{Ab}}((-)^{\mathsf{ab}},-)\simeq\mathrm{Hom}_{\mathsf{Grp}}(-,i(-))\ , \tag{2.43}$$

i.e. abelianization is left adjoint to the inclusion functor.

EXAMPLE 2.5.8. We have a forget functor $i\colon\mathsf{Ab}\to\mathsf{CMon}$ from the category of abelian groups into the category of commutative monoids. We'll construct a left adjoint to this. To this end, we need a way to upgrade a commutative monoid $(M,+)$ to an abelian group. The idea is to formally add negatives—like how you get from $\mathbb{N}$ to $\mathbb{Z}$. For every element $m_+\in M$ you want a formal negative m_- so that $m_++m_-=0$. We will thus have to deal with two kinds of elements: the positives and the negatives. Of course, we can "mix" them by adding, so what we should consider is the product $M\times M$ with the component-wise addition, i.e.

$$(m_+,m_-)+(n_+,n_-)\coloneqq(m_++n_+,m_++n_-)\ . \tag{2.44}$$

This is not yet the whole truth. If you have element (m_+,m_-) and you add a fixed $k\in M$ to both the positive and negative parts, this should clearly still be the same element, i.e. there should be a relation

$$(m_++k,m_-+k)\sim(m_+,m_-)\ .$$

More generally, there should be a relation

$$(m_+,m_-)\sim(n_+,n_-):\Leftrightarrow m_++n_-+k=n_++m_-+k \text{ for some } k\in M\ . \tag{2.45}$$

This is an equivalence relation on $M \times M$. The addition on $M \times M$ is compatible with this relation and therefore descends to an addition on the quotient

$$G(M) \coloneqq (M \times M)/\sim \,, \tag{2.46}$$

making it a commutative monoid. We denote the equivalence class of (m_+, m_-) in $G(M)$ by $[(m_+, m_-)]$. The commutative monoid $G(M)$ is in fact a group since

$$[(m_+, m_-)] + [(m_-, m_+)] = [(m_+ + m_-, m_+ + m_-)] = [(0,0)] \,, \tag{2.47}$$

i.e.

$$-[(m_+, m_-)] = [(m_-, m_+] \,. \tag{2.48}$$

This group is called the **Grothendieck group** of M. We have a canonical map

$$j \colon M \to G(M) \,, \quad m \mapsto [(m, 0)] \,. \tag{2.49}$$

Note that

$$j(m) = j(n) \Leftrightarrow [(m,0)] = [(n,0)] \Leftrightarrow m + k = n + k \text{ for some k} \,, \tag{2.50}$$

and this implies $m = n$ only if M is **cancellative** (which means precisely that you can conclude $m = n$ here). Hence, j is injective, and so M embeds into $G(M)$, if and only if M is cancellative.

Anyways, taking the Grothendieck group yields a functor

$$G(-) \colon \mathsf{CMon} \to \mathsf{Ab} \tag{2.51}$$

when we map a monoid morphism $f \colon M \to M'$ to the group morphism

$$\begin{array}{ccc} G(M) & \overset{G(f)}{\to} & G(M') \\ [(m_+, m_-)] & \mapsto & [f(m_+), f(m_-)] \,. \end{array} \tag{2.52}$$

It is easy to prove that $G \colon \mathsf{CMon} \to \mathsf{Ab}$ is a left adjoint to the forget functor $i \colon \mathsf{Ab} \to \mathsf{CMon}$. I'll leave the final bit of work to you in the following exercise.

Exercise 2.5.9. Prove that the Grothendieck group $G \colon \mathsf{CMon} \to \mathsf{Ab}$ is a left adjoint to the forget functor $i \colon \mathsf{Ab} \to \mathsf{CMon}$.

Example 2.5.10. A functor can have both a left adjoint and a right adjoint. Consider the forget functor $G \colon \mathsf{Top} \to \mathsf{Set}$. As we know from Table 2.1, G has a left adjoint mapping a set X to X equipped with the discrete topology (i.e. every subset of X is open). But G also has a right adjoint mapping X to X with the trivial topology (only $\emptyset$ and X are open).

Let's get back to general theory. I started this section by saying that there's a weaker concept of equivalence—and by that I meant of course adjunctions. But why should they generalize equivalences? Equivalences are about isomorphisms $\varepsilon \colon FG \to \mathrm{id}$ and $\eta \colon \mathrm{id} \to GF$, whereas adjunctions are about isomorphisms

$$\Phi \colon \mathrm{Hom}_{\mathcal{C}'}(F(-), -) \overset{\cong}{\Rightarrow} \mathrm{Hom}_{\mathcal{C}}(-, G(-)) \,. \tag{2.53}$$

Where's the ε and η in an adjunction? We get them as follows. Suppose that (F, G, Φ) is an adjunction between $\mathcal{C}$ and $\mathcal{C}'$. Then for an object $X' \in \mathcal{C}'$ we have an isomorphism

$$\Phi_{G(X'),X'} \colon \mathrm{Hom}_{\mathcal{C}'}(FG(X'), X') \to \mathrm{Hom}_{\mathcal{C}}(G(X'), G(X')) \tag{2.54}$$

and we can thus define

$$\varepsilon_{X'} \coloneqq \Phi^{-1}_{G(X'),X'}(\mathrm{id}_{G(X')}) \in \mathrm{Hom}_{\mathcal{C}'}(FG(X'), X') \,. \tag{2.55}$$

Similarly, for $X \in \mathcal{C}$ we have an isomorphism

$$\Phi_{X,F(X)} \colon \operatorname{Hom}_{\mathcal{C}'}(F(X), F(X)) \to \operatorname{Hom}_{\mathcal{C}}(X, GF(X)) \tag{2.56}$$

and we can thus define

$$\eta_X \coloneqq \Phi_{X,F(X)}(\mathrm{id}_{F(X)}) \in \operatorname{Hom}_{\mathcal{C}}(X, GF(X)) \ . \tag{2.57}$$

By naturality of Φ, the $\varepsilon_{X'}$ and η_X are natural as well, hence they define morphisms

$$\varepsilon \colon FG \to \mathrm{id} \quad \text{and} \quad \eta \colon \mathrm{id} \to GF \ . \tag{2.58}$$

These are called the **counit**, respectively **unit**, of the adjunction (F, G, Φ). There is no way we can conclude from the above that these are isomorphisms—and in general they won't be! But they satisfy a pair of fundamental intertwining equations, called **counit-unit equations**. In the proof of the following lemma it'll be the first time we really use the naturality property of a morphism between functors. Make sure you understand the proof because we'll use similar arguments many more times.

Lemma 2.5.11. *The following equations hold:*

$$\mathrm{id}_F = \varepsilon F \circ F\eta \quad \textit{and} \quad \mathrm{id}_G = G\varepsilon \circ \eta G \ . \tag{2.59}$$

This means that for any $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$ the following equations hold:

$$\mathrm{id}_{F(X)} = \varepsilon_{F(X)} \circ F(\eta_X) \quad \textit{and} \quad \mathrm{id}_{G(X')} = G(\varepsilon_{X'}) \circ \eta_{G(X')} \ . \tag{2.60}$$

Proof. The equations follow from the naturality of Φ. We will first show that

$$\Phi_{X,X'}(f) = G(f) \circ \eta_X \tag{2.61}$$

for any $f \colon F(X) \to X'$. When we then substitute

$$X \rightsquigarrow G(X') \quad \text{and} \quad f \rightsquigarrow \varepsilon_{X'} \colon FG(X') \to X' \ , \tag{2.62}$$

we obtain

$$\mathrm{id}_{G(X')} = \Phi_{G(X'),X'}(\Phi^{-1}_{G(X'),X'}(\mathrm{id}_{G(X')})) = \Phi_{G(X'),X'}(\varepsilon_{X'}) = G(\varepsilon_{X'}) \circ \eta_{G(X')} \ , \tag{2.63}$$

which is the second equation in the claim. To prove (2.61), we use naturality of

$$\Phi_{X,-} \colon \operatorname{Hom}_{\mathcal{C}'}(F(X), -) \to \operatorname{Hom}_{\mathcal{C}}(X, G(-)) \ . \tag{2.64}$$

Applied to $f \colon F(X) \to X'$ this yields the commutativity of the following diagram, which is exactly (2.61):

$$\begin{array}{ccccc}
\mathrm{id}_{F(X)} & \longmapsto & & \longmapsto & \eta_X \\
\Big\downarrow & \operatorname{Hom}_{\mathcal{C}'}(F(X),F(X)) & \xrightarrow{\Phi_{X,F(X)}} & \operatorname{Hom}_{\mathcal{C}}(X,GF(X)) & \Big\downarrow \\
 & {\scriptstyle \operatorname{Hom}_{\mathcal{C}}(F(X),f)}\Big\downarrow & & \Big\downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(X,G(f))} & \\
 & \operatorname{Hom}_{\mathcal{C}'}(F(X),X') & \xrightarrow[\Phi_{X,X'}]{} & \operatorname{Hom}_{\mathcal{C}}(X,G(X')) & \\
f & \longmapsto & & \longmapsto & \begin{array}{c} G(f) \circ \eta_X \\ = \Phi_{X,X'}(f) \end{array}
\end{array} \tag{2.65}$$

With a similar naturality argument, you prove that

$$\Phi^{-1}_{X,X'}(g) = \varepsilon_{X'} \circ F(g) \tag{2.66}$$

for any $g\colon X \to G(X')$, and then substituting

$$X' \rightsquigarrow F(X) \quad \text{and} \quad g \rightsquigarrow \eta_X : X \to GF(X) \tag{2.67}$$

yields the first equation in the claim. □

We have shown that an adjunction (F, G, Φ) induces a counit-unit pair (ε, η) satisfying the counit-unit equations. I'll leave it now to you to prove that if conversely we have a tuple $(F, G, \varepsilon, \eta)$ where $\varepsilon\colon FG \to \mathrm{id}$ and $\eta\colon \mathrm{id} \to GF$ are morphisms satifying the counit-unit equations, then this induces an adjunction (F, G, Φ) with

$$\Phi_{X,X'}(f) \coloneqq G(f) \circ \eta_X \quad \text{and} \quad \Phi^{-1}_{X,X'}(g) \coloneqq \varepsilon_{X'} \circ F(g) \tag{2.68}$$

Moreover, the constructions $\Phi \rightsquigarrow (\varepsilon, \eta)$ and $(\varepsilon, \eta) \rightsquigarrow \Phi$ are inverse to each other so that we can equivalently define adjunctions via a counit-unit.

Exercise 2.5.12. Prove the claims in the paragraph above.

Now, that we have extracted a counit-unit pair from an adjunction, we're in a better position to compare this to an equivalence of categories—where counit and unit are isomorphisms. First, let's introduce the following concept.

Definition 2.5.13. An adjunction $(F, G, \varepsilon, \eta)$ where ε and η are isomorphisms is called an **adjoint equivalence**.

Clearly, an adjoint equivalence is an equivalence. But it is a special equivalence since (ε, η) also satisfy the counit-unit equations, which were not part of the definition of a general equivalence. However, any equivalence can be "modified" into an adjoint equivalence in the following sense.

Lemma 2.5.14. *Let (F, G, η) be (part of) an equivalence, where $\eta\colon \mathrm{id} \to GF$ is an isomorphism. Then there is a unique isomorphism $\varepsilon\colon FG \to \mathrm{id}$ such that $(F, G, \varepsilon, \eta)$ is an adjoint equivalence.*

Proof. Let $\xi\colon FG \to \mathrm{id}$ be an isomorphism such that (F, G, ξ, η) is an equivalence. We're going to modify ξ to an isomorphism ε such that (ε, η) satisfies the counit-unit equations. Namely, we define ε as the composition

$$\underbrace{FG \xrightarrow{FG\xi^{-1}} FGFG \xrightarrow{F\eta^{-1}G} FG \xrightarrow{\xi} \mathrm{id}}_{=:\varepsilon} \tag{2.69}$$

This means explicitly for $X' \in \mathcal{C}'$:

$$\underbrace{FG(X') \xrightarrow{FG(\xi^{-1}_{X'})} FGFG(X') \xrightarrow{F(\eta^{-1}_{G(X')})} FG(X') \xrightarrow{\xi_{X'}} X'}_{=:\varepsilon_{X'}} \tag{2.70}$$

By naturality, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\xi^{-1}_{F(X)}} & FGF(X) \\ {\scriptstyle F(\eta_X)}\downarrow & & \downarrow{\scriptstyle F(\eta_{GF(X)})} \\ FGF(X) & \xrightarrow[FG(\xi^{-1}_{F(X)})]{} & FGFGF(X) \end{array} \tag{2.71}$$

This means $FG(\xi_{F(X)}^{-1}) \circ F(\eta_X) = F(\eta_{GF(X)}) \circ \xi_{F(X)}^{-1}$, hence

$$\underbrace{\xi_{F(X)} \circ F(\eta_{GF(X)}^{-1}) \circ FG(\xi_{F(X)}^{-1})}_{\varepsilon_{F(X)}} \circ F(\eta_X) = \mathrm{id}_{F(X)} \ , \tag{2.72}$$

and that's exactly one of the counit-unit equations. As it's Saturday 10pm, I'll leave it up to you to prove the other equation and show uniqueness of ε. □

Recall from Lemma 2.4.7 that a weak inverse of an equivalence is unique up to isomorphism. We have this property also for adjoints.

LEMMA 2.5.15. *A left (or right) adjoint of a functor is unique up to isomorphism.*

PROOF. We prove the claim for right adjoints—the proof for left adjoints works similarly. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor and let $G, G'\colon \mathcal{C}' \to \mathcal{C}$ be two right adjoints. We then have isomorphisms

$$\mathrm{Hom}_{\mathcal{C}'}(F(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

and

$$\mathrm{Hom}_{\mathcal{C}'}(F(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, G'(-)) \ .$$

Composition yields an isomorphism

$$\eta\colon \mathrm{Hom}_{\mathcal{C}}(-, G(-)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(-, G'(-)) \ .$$

In particular, for fixed $X' \in \mathcal{C}'$ we have an isomorphism

$$\eta_{X'}\colon \mathrm{Hom}_{\mathcal{C}}(-, G(X')) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(-, G'(X')) \ .$$

Now, recall the Yoneda lemma from Example 2.4.12. This states that the functor $h_-\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Set})$ mapping $Y \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ is an embedding, so in particular fully faithful. Since $\mathrm{Hom}_{\mathcal{C}}(-, G(X')) = h_{G(X')}$ and $\mathrm{Hom}_{\mathcal{C}}(-, G'(X')) = h_{G'(X')}$, this means that there is a unique morphism

$$f_{X'}\colon G(X') \to G'(X')$$

with $\eta_{X'} = h_{f_{X'}}$. This morphism is moreover an isomorphism by fully-faithfulness of h_-. Now, you can think a bit and see that the fact that $\eta_{X'}$ is natural in X' implies that $f_{X'}$ is natural in X' as well, hence it yields an isomorphism $G \to G'$. □

CHAPTER 3

Abelian categories

Recall how we arrived at the concept of categories: there are dozens of examples of (algebraic) structures, each coming with their structure preserving maps, and we wanted a general framework to talk about objects and morphisms. One part of category theory is now to try to find categorical generalizations of concepts we know in particular categories or examples. I will call this process **categorization**[1]. Recall what we did when we looked at injective maps: we formulated this categorically as monomorphisms and then studied this in various categories. In this way you introduce special objects and special morphisms, and even special categories where these objects and morphisms behave in the way you want them. The notion of abelian categories that we are going to introduce in this chapter comes precisely from trying to find categorization of important constructions you know from abelian groups (or from vector spaces or modules in general) like the direct sum of abelian groups and the kernel of a morphism. The upshot of this is that if you stumble across a new category and can show that it has the desired properties—like being abelian—you can use all of the general results about such categories to study this particular category without reproving anything that follows from general features anyways.

This is maybe not yet final philosophy. I also don't like the word "categorization".

3.1. Additive categories

The first thing we want to categorize is the direct sum of abelian groups (or vector spaces or modules). Recall that the direct sum of two abelian groups A_1 and A_2 is

$$A_1 \oplus A_2 \coloneqq \{(a_1, a_2) \mid a_i \in A_i\} \,, \tag{3.1}$$

which is again an abelian group with respect to pointwise addition. Unfortunately, this is not a categorical definition because $a_i \in A_i$ does not make sense in a general category. So, let's try to make this categorical. What's the point of the direct sum when you think about morphisms? It's the following: given morphisms $f_i \colon A \to A_i$ *from* a fixed abelian group A, there's a unique morphism $f \colon A \to A_1 \oplus A_2$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A_1 \oplus A_2 \\ & \searrow^{f_i} & \downarrow{\scriptstyle \mathrm{p}_i} \\ & & A_i \end{array} \tag{3.2}$$

commutes, where

$$\mathrm{p}_i \colon A_1 \oplus A_2 \to A \tag{3.3}$$

[1] I don't think there's official terminology. "Categoricalization" or "arrowfication" would also work and describe the process more precisely actually but this all sounds like a nightmare.

is the **projection** onto the i-th component. Namely, you define

$$f(a) \coloneqq (f_1(a), f_2(a)) \ . \tag{3.4}$$

But the direct sum also satisfies a dual property as well: given morphisms $f_i\colon A_i \to A$ *into* an abelian group A, there's a unique morphism $f\colon A_1 \oplus A_2 \to A$ such that the diagram

$$\begin{array}{ccc} A_1 \oplus A_2 & \xrightarrow{f} & A \\ {\scriptstyle \mathrm{i}_i}\uparrow & \nearrow{\scriptstyle f_i} & \\ A_i & & \end{array} \tag{3.5}$$

commutes, where

$$\mathrm{i}_i\colon A_i \to A_1 \oplus A_2 \tag{3.6}$$

is the **inclusion** of the i-th component. Namely, you define

$$f((a_1, a_2)) \coloneqq f_1(a_1) + f_2(a_2) \ . \tag{3.7}$$

Between the projections and embeddings we have the relation

$$\mathrm{p}_i \circ \mathrm{i}_i = \mathrm{id}_{A_i} \ , \quad \mathrm{p}_j \circ \mathrm{i}_i = 0 \text{ for } i \neq j \ . \tag{3.8}$$

The important thing to notice is that the direct sum is not just an object but comes together with morphisms (projection and inclusion) satisfying some special properties. Let us first view the two properties (3.2) and (3.5) separately and formulate them in an arbitrary category.

Definition 3.1.1. Let $\mathcal{C}$ be a category and let $X_1, X_2 \in \mathcal{C}$.

(1) A **product** of X_1 and X_2 is an object $X_1 \times X_2$ together with morphisms $\mathrm{p}_i\colon X_1 \times X_2 \to X_i$ satisfying the following property: given any morphisms $f_i\colon X \to X_i$ there is a unique morphism $f\colon X \to X_1 \times X_2$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X_1 \times X_2 \\ & {\scriptstyle f_i}\searrow & \downarrow{\scriptstyle \mathrm{p}_i} \\ & & X_i \end{array} \tag{3.9}$$

commutative.

(2) A **coproduct** of X_1 and X_2 is an object $X_1 \coprod X_2$ together with morphisms $\mathrm{i}_i\colon X_i \to X_1 \coprod X_2$ satisfying the following property: given any morphisms $f_i\colon X_i \to X$ there is a unique morphism $f\colon X_1 \coprod X_2 \to X$ making the diagram

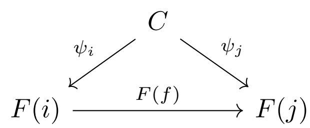
$$\begin{array}{ccc} X_1 \coprod X_2 & \xrightarrow{f} & X \\ {\scriptstyle \mathrm{i}_i}\uparrow & \nearrow{\scriptstyle f_i} & \\ X_i & & \end{array} \tag{3.10}$$

commutative.

Before we look into this more closely, let's look at this more generally. These definitions are always about an object together with morphisms satisfying some particular property. Many categorizations of concepts we know from abelian groups go exactly along these lines. Therefore, it makes sense to generalize this concept immediately.

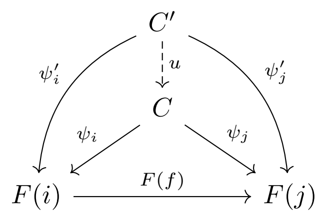
Definition 3.1.2. Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor. A **cone** to F is an object $C \in \mathcal{C}$ together with a family ψ of morphisms $\psi_i\colon C \to F(i)$ for any $i \in \mathcal{I}$ such that for every morphism $f\colon i \to j$ in $\mathcal{I}$ the diagram

It's maybe better to use D instead of F here because of later context.



(3.11)

commutes.[2] A **limit** (or **universal cone**) to F is a cone (C, ψ) to F such that for any other cone (C', ψ') to F there is a *unique* morphism $u\colon C' \to C$ making the diagram

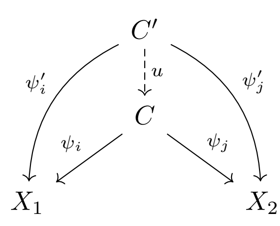


(3.12)

commutative for all morphisms $f\colon i \to j$ in $\mathcal{C}$.[3]

You should think of the functor $F\colon \mathcal{I} \to \mathcal{C}$ as a **diagram** in $\mathcal{C}$ of **shape** $\mathcal{I}$, i.e. a collection of objects and morphisms in $\mathcal{C}$ indexed by $\mathcal{I}$. Often, people forget about the morphisms coming along with the limit but they are there and they are important. I leave it up to you to formulate the dual concept of a **co-cone** and a **colimit**.

Example 3.1.3. We can consider any set (even class) S as a category whose objects are the elements of S and whose morphisms are only the identity morphisms. Such categories are called **discrete**. Let's consider a two-element set $\{1, 2\}$. Then a functor $F\colon \{1, 2\} \to \mathcal{C}$ is just a choice of two objects $X_1 \coloneqq F(1)$ and $X_2 \coloneqq F(2)$ in $\mathcal{C}$. Now, what is a cone to F? This is an object $C \in \mathcal{C}$ together with morphisms $\psi_i\colon C \to X_i$. There's no further condition because there are no (non-identity) morphisms in the discrete category $\{1, 2\}$. Now, C is universal if for any object $C' \in \mathcal{C}$ together with morphisms $\psi'_i\colon C' \to X_i$ there is a unique morphism $u\colon C' \to C$ making the diagram



(3.13)

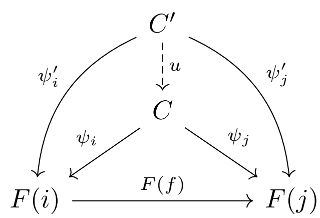
commutative. Do you realize that the universal cone satisfies exactly the defining property of the product of X_1 and X_2? Similarly, the coproduct is precisely the colimit of a corresponding functor $\{1, 2\} \to \mathcal{C}$. Now, you can easily define products and coproducts of arbitrary families of objects as well: it's just the limit, respectively colimit, of a diagram whose shape is a discrete category.

[2] Can you see why (C, ψ) is called a "cone"?

[3] So, another way to say is that any other cone to F "factors through" the universal cone.

Another upshot of the limit point of view is that it's basically straightforward to see that if a diagram has a limit, then it is unique up to unique isomorphism. This tells us for example that the property of product and coproduct that we extracted from what we know about abelian groups is indeed a characterizing property and we're on the right track with Definition 3.1.1.

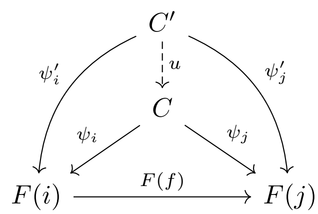
LEMMA 3.1.4. *A limit (C, ψ) of a diagram $F: \mathcal{I} \to \mathcal{C}$ is unique up to unique isomorphism, i.e. if (C', ψ') is another limit, then there is a unique isomorphism $u: C' \to C$ making the diagram*



(3.14)

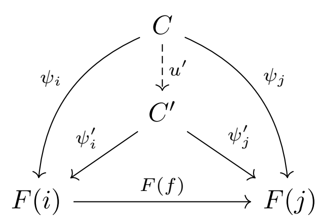
commutative for all morphisms $f: i \to j$ in $\mathcal{C}$.

PROOF. The idea is simply to mutually apply the universal property. Since (C, ψ) is a universal cone and (C', ψ') is another cone, there is a morphism $u: C' \to C$ making the diagram



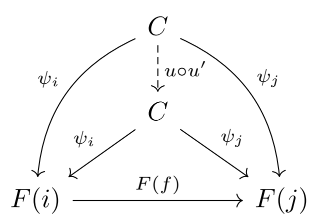
(3.15)

commutative. Conversely, since (C', ψ') is a universal cone and (C, ψ) is another cone, there is a morphism $u': C \to C'$ making the diagram



(3.16)

commutative. When we stack the last diagram onto the previous, we obtain a commutative diagram



(3.17)

Now, the universal property of C tells us that there is a unique morphism $C \to C$ making this diagram commutative. The identity obviously does, so we conclude that $u \circ u' = \mathrm{id}_C$. Analogously, we conclude that $u' \circ u = \mathrm{id}_{C'}$. Hence, u is an isomorphism and the uniqueness is clear as well. □

DEFINITION 3.1.5. One also writes $\lim F$ for the limit of F and $\operatorname{colim} F$ for the colimit (if it exists).

Now that we know that the product is just a special case of a limit and that limits are unique, it's time to ask the key question: does a limit actually exist? In general, the answer is: no. For a given shape $\mathcal{I}$, this is really a condition on the category $\mathcal{C}$. If $\mathcal{C}$ has a limit for any diagram of shape $\mathcal{I}$ we say that $\mathcal{C}$ **has $\mathcal{I}$-limits**. If $\mathcal{C}$ has $\mathcal{I}$-limits for any small category $\mathcal{I}$ one says that $\mathcal{C}$ is **complete**. Dually, one defines **cocomplete**, and a category that is both complete and cocomplete is called **bicomplete.** There's an existence theorem for limits which basically says that if the category has (arbitrary) products and moreover limits of one special simple shape (equalizers), it's already complete. There would be a lot to say about this but we won't go down this rabbit hole of abstract nonsense and just look at a few examples.

I should actually prove this here.

EXAMPLE 3.1.6. The category Set is bicomplete. For example, the product is the Cartesian product and the coproduct is the disjoint union.

EXAMPLE 3.1.7. The category Ab is bicomplete. The product of a family $(A_i)_{i\in I}$ of abelian groups is the Cartesian product

$$\prod_{i\in I} A_i := \{(a_i)_{i\in I} \mid a_i \in A_i\}$$

with component-wise addition. The coproduct on the other hand is the direct sum

$$\bigoplus_{i\in I} A_i := \{(a_i)_{i\in I} \mid a_i \in A_i, \text{all but finitely many } a_i = 0\} .$$

You see that for finite I, the product and coproduct are the same; but for infinite I they are distinct. You have the analogous constructions in the category R-Mod of R-modules over a ring R.

EXAMPLE 3.1.8. The category Grp of groups is bicomplete. The product of a family of groups is just the Cartesian product of the underlying sets equipped with component-wise multiplication. The coproduct, however, is given by the so-called free product, and this looks quite different. Let's not discuss this here.

EXAMPLE 3.1.9. Finally, a non-example. The category $\mathcal{C}$ of fields (full subcategory of CRing consisting of fields) does not have products—not even finite ones. Let K and L be fields. A product $K \times L$ would need to have projection maps $K \times L \to K$ and $K \times L \to L$. But if K and L have different characteristic (e.g. $K = \mathbb{Q}$ and $L = \mathbb{F}_2$) such morphisms just cannot exist. For the exact same reason, coproducts do not exist either.

After this detour, let's come back to what we initially wanted to categorize: the direct sum of two abelian groups. From the discussion it's clear that the direct sum is both a product and a coproduct at the same time. But that's not all: there's this intertwining relation (3.8). This is still a bit of an issue since in the relation $\mathrm{p}_j \circ \mathrm{i}_i = 0$ for $i \neq j$ the 0 is the zero morphism $A_i \to A_j$. In a general category there is no zero morphism, so we first need to categorize this. Maybe, before thinking about the zero *morphism*, let's think about the zero *object* which should categorize the zero abelian group. How can we characterize this? Well, there's a unique morphism $0 \to A$ into every abelian group and a unique morphism $A \to 0$ from every abelian group. Maybe this is enough, so let's define:

DEFINITION 3.1.10. Let $\mathcal{C}$ be a category.

(1) An object I is **initial** if there is a unique morphism $I \to X$ into any X.
(2) An object T is **terminal** if there is a unique morphism $X \to T$ for any X.
(3) A **zero object** is an object 0 which is both initial and terminal.

Can define this as limit/colimit. Uniqueness then clear.

LEMMA 3.1.11. *If $\mathcal{C}$ has an initial (resp. terminal, zero) object, then it is unique up to unique isomorphism.*

PROOF. We just consider the case of initial objects, the other cases follow analogously. Let I' be another initial object. Then there is a unique morphism $u \colon I' \to I$ and a unique morphism $u' \colon I \to I'$. We thus get a morphism $u \circ u' \colon I \to I$. But also the identity $I \to I$ is a morphism and since there is only one such morphism, we must have $u \circ u' = \mathrm{id}$. Analogously, you prove that $u' \circ u = \mathrm{id}$. Hence, u is an isomorphism and it's clear that this is unique. □

EXAMPLE 3.1.12. In Ab the zero abelian group is the zero object. More generally, in R-Mod the zero module is a zero object.

EXAMPLE 3.1.13. In Grp the trivial group 1 is a zero object.

EXAMPLE 3.1.14. In Set the empty set $\emptyset$ is an initial object and a singleton set $\star$ is a terminal object. It follows that Set does not have a zero object.

EXAMPLE 3.1.15. In Ring the ring of integers $\mathbb{Z}$ is an initial object and the zero ring 0 is a terminal object. In particular, Ring does not have a zero object.

To categorize the direct sum we need a zero *morphism*, not really a zero *object*. Where do we get this from? When we have a zero object 0, we get a special morphism between any two objects X and Y, namely

$$X \longrightarrow 0 \longrightarrow Y, \qquad \underbrace{X \to Y}_{=:0_{XY}} \tag{3.18}$$

where the two morphisms are the unique ones we get from the fact that 0 is a zero object. We call the resulting morphism the **zero morphism** from X to Y. There's one little problem with this definition: a zero object is only unique up to isomorphism and therefore the zero morphism may—in principle—depend on the choice of the zero object. But it does not: if $0'$ is another zero object then there is a unique morphism $0 \to 0'$ and the resulting morphism

$$X \to 0 \to 0' \to Y$$

must be both 0_{XY} and $0'_{XY}$. In other words, there is only one morphism $X \to Y$ factoring through a zero object, and this is the zero morphism.

EXAMPLE 3.1.16. Given two abelian groups (or more generally modules) A and B the unique morphism $A \to 0$ is $a \mapsto 0$ and the unique morphism $0 \to B$ is $0 \mapsto 0$. Hence, the zero morphism $A \to B$ is $a \mapsto 0$. Great!

Now, we can finally categorize the direct sum of abelian groups.

Definition 3.1.17. Let $\mathcal{C}$ be a category with zero object.[4] The **direct sum** (or **biproduct**) of two objects X_1 and X_2 is an object $X_1 \oplus X_2$ together with morphisms $\mathrm{p}_i \colon X_1 \oplus X_2 \to X_i$ and $\mathrm{i}_i \colon X_i \to X_1 \oplus X_2$ such that:

(1) $X_1 \oplus X_2$ together with the p_i is a product of X_1 and X_2;
(2) $X_1 \oplus X_2$ together with the i_i is a coproduct of X_1 and X_2;
(3) $\mathrm{p}_i \circ \mathrm{i}_i = \mathrm{id}_{X_i}$ and $\mathrm{p}_j \circ \mathrm{i}_i = 0_{X_i X_j}$ for $i \neq j$.

I leave it up to you to generalize the definition to arbitrarily many summands and to show that the direct sum—if it exists—is unique up to unique isomorphism.

Definition 3.1.18. A category is **semiadditive** if it has finite direct sums.[5]

Example 3.1.19. The category R-Mod is semiadditive. The direct sum is precisely as in (3.1) together with the projection and inclusion. In fact, R-Mod even has direct sums of arbitrarily many summands.

Example 3.1.20. The category Grp of groups is not semiadditive. Even though it has a zero object by Example 3.1.13 and also products and coproducts by Example 3.1.8, product and coproduct are distinct, so we cannot define a direct sum.

Exercise 3.1.21. Show that the empty direct sum is the zero object.

Exercise 3.1.22. Show that the direct sum is associative and commutative, i.e. there are canonical isomorphisms

$$(X_1 \oplus X_2) \oplus X_3 \simeq X_1 \oplus (X_2 \oplus X_3)\ , \quad X_1 \oplus X_2 \simeq X_2 \oplus X_1\ . \tag{3.19}$$

Conclude that if $\mathcal{C}$ is semiadditive and essentially small, the set $[\mathcal{C}]$ of isomorphism classes of $\mathcal{C}$ is a commutative monoid with addition

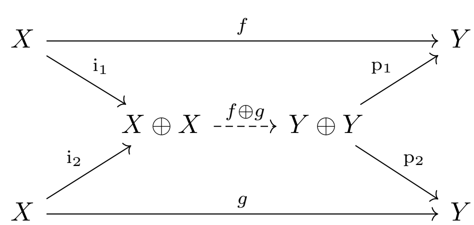
$$[X] + [Y] \coloneqq [X \oplus Y]\ . \tag{3.20}$$

What is the neutral element?

Let $\mathcal{C}$ be a semiadditive category. We can use the direct sum on *objects* to define a direct sum on *morphisms* as well. Namely, if $f, g \colon X \to Y$ are two morphisms, then by the universal property of the direct sum $X \oplus Y$ there is a unique morphism

$$f \oplus g \colon X \oplus X \to Y \oplus Y \tag{3.21}$$

making the diagram



(3.22)

commutative.

Example 3.1.23. In R-Mod the direct sum $f \oplus g$ of two morphisms $f, g \colon A \to B$ is simply the map $(a_1, a_2) \mapsto (f(a_1), g(a_2))$, i.e. it is f in the first component and g in the second component.

[4]Such categories are sometimes also called **pointed**.
[5]For this to make sense we of course require $\mathcal{C}$ to have a zero object.

There's a brilliant way to get from X to $X \oplus X$ and from $Y \oplus Y$ back to Y that allows us to produce from the direct sum $f \oplus g \colon X \oplus X \to Y \oplus Y$ an actual sum $f + g \colon X \to Y$. Namely, the product property of the direct sum $X \oplus X$ yields a **diagonal morphism** $\Delta_X \colon X \to X \oplus X$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \oplus X \\ & \searrow^{\mathrm{id}_X} & \downarrow \mathrm{p}_i \\ & & X \end{array} \tag{3.23}$$

commutative, and dually by the coproduct property there is a **codiagonal morphism** $\nabla_Y \colon Y \oplus Y \to Y$ making the diagram

$$\begin{array}{ccc} Y \oplus Y & \xrightarrow{\nabla_Y} & Y \\ \mathrm{i}_i \uparrow & \nearrow_{\mathrm{id}_Y} & \\ Y & & \end{array} \tag{3.24}$$

commutative. We thus define

$$f + g \coloneqq \nabla_Y \circ (f \oplus g) \circ \Delta_X \colon X \to Y \ . \tag{3.25}$$

Example 3.1.24. In R-Mod the diagonal morphism Δ_A is the map $a \mapsto (a, a)$ and the codiagonal morphism ∇_B is the map $(b, b) \mapsto b + b$. Hence, the sum $f + g$ of two morphisms $f, g \colon A \to B$ is the map $a \mapsto f(a) + g(a)$, i.e. this is simply the pointwise addition of maps—what else would you expect?

Exercise 3.1.25. Show that the addition in (3.25) makes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ into a commutative monoid with neutral element the zero morphism 0_{XY}. Moreover, show that the addition is compatible with the composition.

Exercise 3.1.26. Show that taking the direct sum defines a bifunctor

$$- \oplus - \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \ . \tag{3.26}$$

The addition on morphisms allows us to introduce the following **matrix calculus** for morphisms. Consider a direct sum $X \coloneqq \bigoplus_{i=1}^m X_i$. Let $\mathrm{p}_i \colon X \to X_i$ be the projection and let $\mathrm{i}_i \colon X_i \to X$ be the inclusion. We claim that

$$\sum_{i=1}^m \mathrm{i}_i \circ \mathrm{p}_i = \mathrm{id}_X \ . \tag{3.27}$$

Let u be the sum. Then composition with the inclusion i_j yields

$$u \circ \mathrm{i}_j = \left(\sum_{i=1}^m \mathrm{i}_i \circ \mathrm{p}_i \right) \circ \mathrm{i}_j = \sum_{i=1}^m \mathrm{i}_i \circ \mathrm{p}_i \circ \mathrm{i}_j = \sum_{i=1}^m \mathrm{i}_i \circ \delta_{ij} \, \mathrm{id}_{X_j} = \mathrm{i}_j \ . \tag{3.28}$$

Here, we have used the fact that the addition is associative and compatible with composition (Exercise 3.1.25), together with the equation $\mathrm{p}_j \circ \mathrm{i}_j = \mathrm{id}_{X_j}$ from the properties of the direct sum (Definition 3.1.17). The above equation implies $u = \mathrm{id}_X$ since this is the unique morphism $X \to X$ satisfying this equation by the coproduct property.

Now, take another object $Y \coloneqq \bigoplus_{j=1}^n Y_j$ and consider a morphism $f \colon X \to Y$. Let $\mathrm{q}_j \colon Y \to Y_j$ be the projection and let $\mathrm{j}_j \colon Y_j \to Y$ be the inclusion. Then for any pair (i, j) we obtain the **component** morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{i}_i}\uparrow & & \downarrow{\scriptstyle \mathrm{q}_j} \\ X_i & \underset{=:f_{ji}}{\longrightarrow} & Y_j \end{array} \tag{3.29}$$

We write the component morphisms into a matrix

$$M_f := (f_{ji})_{ij} \ . \tag{3.30}$$

We first note that we can recover f from the matrix M_f because

$$\begin{aligned} \sum_{i,j} \mathrm{j}_j \circ f_{ji} \circ \mathrm{p}_i &= \sum_{ij} \mathrm{j}_j \circ (\mathrm{q}_j \circ f \circ \mathrm{i}_i) \circ \mathrm{p}_i = \sum_{ij} (\mathrm{j}_j \circ \mathrm{q}_j) \circ f \circ (\mathrm{i}_i \circ \mathrm{p}_i) \\ &= \left(\sum_j \mathrm{j}_j \circ \mathrm{q}_j \right) \circ f \circ \left(\sum_i \mathrm{i}_i \circ \mathrm{p}_i \right) = \mathrm{id}_Y \circ f \circ \mathrm{id}_X = f \ , \end{aligned}$$

where we have used the fundamental equation (3.27). The conclusion is that the matrix of a morphism completely determines the morphism.

EXERCISE 3.1.27. Show that addition and composition of morphisms in $\mathcal{C}$ translates into addition and multiplication of the associated matrices, i.e.

$$M_{f+g} = M_f + M_g \quad \text{and} \quad M_{g \circ f} = M_g M_f \ . \tag{3.31}$$

Recall the addition of morphisms in R-Mod from Example 3.1.24. In this case, the addition also has negatives so that the Hom-sets are not just commutative *monoids* but abelian *groups*—in fact they are R-modules naturally. It'll be useful to have a general concept of categories whose Hom-sets have a compatible module structure.

DEFINITION 3.1.28. Let R be a commutative ring. An **R-linear structure** on a category $\mathcal{C}$ consists of an R-module structure on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any two objects X and Y such that the composition

$$\circ\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z) \tag{3.32}$$

is R-bilinear.[6] A category is said to be **R-linear** if it admits an R-linear structure.[7].

DEFINITION 3.1.29. A $\mathbb{Z}$-linear category is also said to be **preadditive**.

Clearly, an R-linear category is naturally preadditive.

EXAMPLE 3.1.30. R-Mod has a natural R-linear structure given by point-wise addition of morphisms.

EXERCISE 3.1.31. Show that in an R-linear category $\mathcal{C}$ the set $\mathrm{End}_{\mathcal{C}}(X)$ of endomorphisms of an object X is naturally an R-algebra with respect to the composition as multiplication.

[6]We assume that R is *commutative* since otherwise for the bilinearity we would need $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ to be a *right* R-module and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to be a *left* R-module, so actually the Hom-sets would need to be R-bimodules (which they are naturally if R is commutative). You can do all this indeed, but we don't need this generality here.

[7]When one says "Let $\mathcal{C}$ be an R-linear category." it is understood that one fixes an R-linear structure on $\mathcal{C}$.

Example 3.1.32. The category Grp of groups is not preadditive. You can find a neat argument on https://en.wikipedia.org/wiki/Category_of_groups.

Add this argument here.

Now, here's something that requires some thought. Suppose we have a category $\mathcal{C}$ that is semiadditive and also has a preadditive structure. Then we actually have *two* additions on the Hom-sets: the one from the preadditive structure and the one induced by semiadditivity as in (3.25). Are these two additions distinct? Interestingly, the answer is: no, they are identical! In other words:

Lemma 3.1.33. *There is at most one preadditive structure on a semiadditive category.*

Proof. Let $\mathcal{C}$ be a semiadditive category. Suppose that $\mathcal{C}$ has a preadditive structure and denote by $\dotplus$ the corresponding addition on morphisms. We want to show that $f \dotplus g = f + g$ for any morphisms $f, g \colon X \to Y$, where $+$ is the addition (3.25) from the semiadditive structure. In the exact same way as discussed above, we can introduce a matrix calculus with respect to $\dotplus$. This is because all we used was that $+$ is associative and compatible with the composition—and $\dotplus$ satisfies this as well. The associated matrices are independent of the addition because this is not involved in their definition. By (3.25) we have

$$f + g = \nabla_Y \circ (f \oplus g) \circ \Delta_X \,. \tag{3.33}$$

Let's look at the associated matrix M of $\nabla_Y \circ (f \oplus g) \circ \Delta_X$. We have

$$M_{\Delta_X} = \begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} \,, \quad M_{f \oplus g} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \,, \quad M_{\nabla_Y} = \begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix} \,. \tag{3.34}$$

Hence, using the matrix calculus with respect to $\dotplus$, we get

$$M = M_{\nabla_Y} M_{f \oplus g} M_{\Delta_X} = \begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} = \begin{pmatrix} f \dotplus g \end{pmatrix} \,, \tag{3.35}$$

i.e. $f + g = f \dotplus g$ as claimed. □

Now, we finally come to the key definition of this section:

Definition 3.1.34. A semiadditive category which has a preadditive structure is called **additive**.

Note that Lemma 3.1.33 really tells us that a semiadditive category is additive if and only if the addition (3.25) on morphisms from the semiadditive structure has negatives. In particular, additivity is a *property* of a category, not a *structure*.

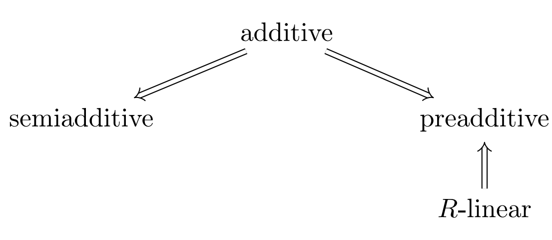


Figure 3.1. *Properties* of categories (semiadditive and additive) and *structures* on categories (preadditive and R-linear) discussed so far.

EXAMPLE 3.1.35. The category R-Mod is additive.

Our motivation for developing a general concept of additive categories was the category of abelian groups (and more generally modules). So far, this was our only example. This whole journey would be completely pointless if there would be no other examples. But luckily there are plenty!

EXAMPLE 3.1.36. Let $\mathcal{C}$ be an additive category, e.g. $\mathcal{C} = R$-Mod. A **graded object** over $\mathcal{C}$ is a sequence $A_\bullet := (A_i)_{i\in\mathbb{Z}}$ of objects $A_i \in \mathcal{A}$. A **morphism** of graded objects $A_\bullet \to B_\bullet$ is a sequence $f_\bullet := (f_i)_{i\in\mathbb{Z}}$ of morphisms $f_i \colon A_i \to B_i$. Graded objects together with morphisms of graded objects form a category $\mathsf{Gr}(\mathcal{C})$. It's not hard to see that $\mathsf{Gr}(\mathcal{C})$ is an additive category: you just define the direct sum of objects and the sum of morphisms component-wise. What we defined should be more precisely called $\mathbb{Z}$-graded objects—you can consider gradings more generally over an arbitrary index set. Note that we have in particular defined the notions of graded vector spaces and graded modules.

EXAMPLE 3.1.37. The following example is an enhancement of graded objects and occurs everywhere in nature. Fix again an additive category $\mathcal{C}$. A **chain complex** over $\mathcal{C}$ is a sequence $A_\bullet := (A_i, d_i)_{i\in\mathbb{Z}}$ of objects $A_i \in \mathcal{C}$ and morphisms $A_{i-1} \xleftarrow{d_i} A_i$ called **differentials** with the property that the composition of two differentials is zero:

$$A_{i-2} \xleftarrow{d_{i-1}} A_{i-1} \xleftarrow{d_i} A_i \,, \qquad d_{i-1}\circ d_i = 0 \tag{3.36}$$

A **morphism** $A_\bullet \to B_\bullet$ of chain complexes is a sequence $f_\bullet := (f_i)_{i\in\mathbb{Z}}$ of morphisms $f_i \colon A_i \to B_i$ such that for each i the diagram

$$\begin{array}{ccc} A_{i-1} & \xleftarrow{d_{A,i}} & A_i \\ {\scriptstyle f_{i-1}}\downarrow & & \downarrow{\scriptstyle f_i} \\ B_{i-1} & \xleftarrow[d_{B,i}]{} & B_i \end{array} \tag{3.37}$$

commutes. Chain complexes over $\mathcal{C}$ together with morphisms of chain complexes form a category that we will denote by $\mathsf{Ch}_\bullet(\mathcal{C})$. Again, $\mathsf{Ch}_\bullet(\mathcal{C})$ is an additive category by defining the direct sum of objects and the sum of morphisms component-wise. There's a dual notion of a **cochain complex** which goes the other way around: this is a sequence $A^\bullet := (A^i, d^i)_{i\in\mathbb{Z}}$ with $d^i \colon A^i \to A^{i+1}$ satisfying $d^{i+1} \circ d^i = 0$, and you get an additive category $\mathsf{Ch}^\bullet(\mathcal{C})$. You may wonder about the strange differential-square condition (3.36). We'll come to this later.

EXAMPLE 3.1.38. The category $\mathsf{Fun}(\mathcal{C}, \mathcal{C}')$ of functors from a category $\mathcal{C}$ into an additive category $\mathcal{C}'$ is additive: the direct sum of objects and the sum of morphisms is defined point-wise.

REMARK 3.1.39. Recall that we introduced zero morphisms by assuming there's a zero object. In a preadditive category you do not necessarily have a zero object but you still have a notion of zero morphisms, namely the zero elements in the Hom-groups. There's a more general notion of zero morphisms generalizing these two cases.

EXERCISE 3.1.40. Let $\mathcal{C}$ be a preadditive category and let $X_1, X_2 \in \mathcal{C}$. Suppose there is $X \in \mathcal{C}$ together with morphisms $\mathrm{p}_i\colon X \to X_i$ and $\mathrm{i}_i\colon X_i \to X$ such that

$$\mathrm{p}_i \circ \mathrm{i}_i = \mathrm{id}_{X_i} \ , \quad \mathrm{p}_j \circ \mathrm{i}_i = 0 \text{ for } i \neq j \ , \quad \mathrm{i}_1 \circ \mathrm{p}_1 + \mathrm{i}_2 \circ \mathrm{p}_2 = \mathrm{id}_X \ . \tag{3.38}$$

Show that $(X, \mathrm{p}_i, \mathrm{i}_i)$ is a direct sum of X_1 and X_2. This can of course be formulated analogously for finitely many summands.

EXERCISE 3.1.41. Let $\mathcal{C}$ be a preadditive category. Show that any finite product (or coproduct) can be completed to a biproduct. Hint: You can construct the inclusions from the universal property of the product applied to specific morphisms.

Now that we have sorted out some special categories, it's time to sort out the appropriate functors as well. There are two things to consider:

(1) functors preserving the direct sum (when we have semiadditive categories);
(2) functors preserving the addition on morphisms (when we have preadditive categories).

Let's start with the first one.

Preservation of direct sums is a special case of preservation of limits and colimits. Let's make this precise. Let $D\colon \mathcal{I} \to \mathcal{C}$ be a functor, interpreted as a diagram, and let (C, ψ) be a cone to D. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor into some other category $\mathcal{C}'$. Then we transfer our diagram D to $\mathcal{C}'$ via $D' := F \circ D\colon \mathcal{I} \to \mathcal{C}'$, and by functoriality $(F(C), F(\psi))$ is a cone to D'. We say that the functor F **preserves** the limit to D if whenever (C, ψ) is a limit (i.e. a universal cone) to D, then $(F(C), F(\psi))$ is a limit to D'. Since limits are unique, it is sufficient to check this for one specific limit. If D does not have a limit, there's no condition. It should be clear what we mean by saying that F preserves $\mathcal{I}$-limits, e.g. all products etc. If F preserves $\mathcal{I}$-limits for all small categories $\mathcal{I}$, then it's said to be **continuous**. The dual formulations for preservation of colimits and for **cocontinuous** functors should be clear.

EXAMPLE 3.1.42. In Ab, the direct product is the Cartesian product as sets equipped with component-wise addition. Hence, the forget functor $\mathsf{Ab} \to \mathsf{Set}$ preserves products. In fact, one can show that this is a continuous functor. This holds more generally for all the forget functors to Set for the categories in Table 2.1. Even more generally, one can prove that every functor having a right adjoint (such as these forget functors) is continuous, and every functor having a left adjoint is cocontinuous. But note that e.g. $\mathsf{Ab} \to \mathsf{Set}$ does not preserve *co*products: the coproduct in Ab is given by the direct sum but in Set it's the disjoint union.

EXERCISE 3.1.43. Let $\mathcal{C}$ be a category. Show that for any object $X \in \mathcal{C}$ the functor $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathsf{Set}$ is continuous. Note: this is basically just the definition of a limit and it's easier than it sounds—but only after you stare at it for a bit! Similarly, for $Y \in \mathcal{C}$ you can (but don't have to because it's analogous) show that the functor $\mathrm{Hom}_{\mathcal{C}}(-, Y)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is continuous. But note that this means that $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ considered as a *contra*variant functor $\mathcal{C} \to \mathsf{Set}$ maps *co*limits to limits!

The direct sum is a bit more special since it's a limit and a colimit simultaneously. But if you have a direct sum $(X_1 \oplus X_2, \mathrm{p}_i, \mathrm{i}_i)$ in $\mathcal{C}$ and you have a functor $F\colon \mathcal{C} \to \mathcal{C}'$, then you get a datum $(F(X_1 \oplus X_2), F(\mathrm{p}_i), F(\mathrm{i}_i))$, and you want this to be the direct sum of $F(X_1)$ and $F(X_2)$. If this holds, you say that F **preserves** the direct sum of X_1 and X_2. You can formulate this of course more generally for direct sums with arbitrarily many summands. A functor preserves finite direct sums if and only if it preserves direct sums of two objects.

Before we look at an example, let's first look at functors preserving the addition on morphisms.

DEFINITION 3.1.44. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between R-linear categories is said to be **R-linear** if the induced local maps

$$F_{X,Y}\colon \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}'}(F(X),F(Y)) \tag{3.39}$$

are R-module morphisms. In case of preadditive categories, i.e. $R = \mathbb{Z}$, one speaks of an **additive** functor.[8]

Here's a nice lemma that is very helpful in practice and connects the two additivity concepts of functors that we introduced.

LEMMA 3.1.45. *A functor between additive categories is additive if and only if it preserves finite direct sums.*

PROOF. Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories. Suppose that $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor. Let $X_1, X_2 \in \mathcal{C}$ and let $(X_1 \oplus X_2, \mathrm{p}_i, \mathrm{i}_i)$ be the direct sum. Because F is additive, we get in the image a tuple satisfying all the conditions of a direct sum in Exercise 3.1.40. Hence the image is the direct sum and this shows that F preserves finite direct sums. Conversely, suppose that F preserves direct sums. Recall from (3.25) that the sum of two morphisms $f, g\colon X \to Y$ is given by

$$f + g = \nabla_Y \circ (f \oplus g) \circ \Delta_X \ .$$

You can easily check that because F preserves direct sums, it maps the diagonal morphism, codiagonal morphism, and direct sum of morphisms to the corresponding constructions in the image, and therefore $F(f+g) = F(f) + F(g)$. □

EXAMPLE 3.1.46. Since R-Mod is R-linear, it follows that the Hom-functor $\operatorname{Hom}_{R\text{-}\mathsf{Mod}}(X,-)\colon R\text{-}\mathsf{Mod} \to \mathsf{Set}$ is actually a functor $R\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$. This functor is R-linear, hence it preserves finite direct sums. You could (and should) also prove this explicitly. In general though, it will not preserve *infinite* direct sums.

Let's think about the appropriate notion of *sub*category for additive categories.

DEFINITION 3.1.47. An **additive subcategory** of an additive category $\mathcal{C}'$ is a subcategory $\mathcal{C}$ of $\mathcal{C}'$ such that:

(1) $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is a subgroup of $\operatorname{Hom}_{\mathcal{C}'}(X,Y)$ for all $X, Y \in \mathcal{C}$;
(2) $\mathcal{C}$ is closed under finite direct sums in $\mathcal{C}'$, i.e. the direct sum in $\mathcal{C}'$ of finitely many objects in $\mathcal{C}$ is also a direct sum in $\mathcal{C}$.

In this case, the category $\mathcal{C}$ is itself additive and the natural functor $\mathcal{C} \to \mathcal{C}'$ is additive. For the second condition, it is clearly enough to check that the zero object (the empty direct sum) of $\mathcal{C}'$ is contained in $\mathcal{C}$ and that $\mathcal{C}$ is closed under direct sums of *two* objects. If $\mathcal{C}$ is a *full* subcategory, then it is sufficient to check that the direct sum *as an object* is contained in $\mathcal{C}$; but if $\mathcal{C}$ is not full, you need to check that it is really a direct sum in the subcategory $\mathcal{C}$ as well because the morphisms you get in $\mathcal{C}'$ from the universal property of the direct sum may not be contained in $\mathcal{C}$.

[8] The last one may sound a bit confusing: *pre*additive categories but additive functors. But there is no *pre*additive functor: you just want an additive map between the Hom-groups—and this is called additive.

EXAMPLE 3.1.48. The category R-mod of finitely generated modules is an additive full subcategory of R-Mod.

EXERCISE 3.1.49. Show that if $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence and $\mathcal{C}$ is additive, then $\mathcal{C}'$ is additive as well and F is additive.

3.2. Abelian categories

Now that we have categorized the direct sum of abelian groups (more generally, of modules), it's time to turn to some further important constructions with modules and try to categorize them as well. Especially, for modules we have a notion of submodules, we can take quotients by submodules, every morphism has a kernel and an image, we have the isomorphism theorems, etc. Categorizing all these concepts, we'll arrive at the notion of abelian categories—categories which in many aspects behave like the category of modules over a ring.

Let's start with kernels. Any morphism $f\colon A \to B$ of modules has a kernel

$$\mathrm{Ker}(f) \coloneqq \{a \in A \mid f(a) = 0\} . \tag{3.40}$$

Your mind is probably quite categorical already and you have noticed that this is not a categorical definition since it uses elements. How do we categorize the kernel? Let $K \coloneqq \mathrm{Ker}(f)$. Then the composition of the inclusion $k\colon \mathrm{Ker}(f) \to A$ with f is the zero morphism $0\colon K \to B$. This is in fact universal since if $k'\colon K' \to A$ is any other morphism with the property that $f \circ k' = 0$, then k' must map into K, i.e. it factorizes through k. Note that this is actually the property of a limit, namely the limit of the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array} \tag{3.41}$$

Observe that (K, k) is really a cone to this diagram—the morphism into B and into 0 are forced so that we drop them in the notation—and the factorization property noted above precisely means that (K, k) is a universal cone, i.e. a limit. That's enough evidence to make a general definition!

DEFINITION 3.2.1. Let $\mathcal{C}$ be a category with zero object. The **kernel** of a morphism $f\colon X \to Y$ is the limit to the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \tag{3.42}$$

As above, note that the morphisms into Y and 0 are forced, so we can think of a kernel as a morphism $k\colon K \to X$ which is universal with respect to the property that $f \circ k = 0$. As for any other limit, if the kernel exists then it is unique—but there's no guarantee it actually exists. We'll denote the kernel as usual by $\mathrm{Ker}(f)$ but note that this is understood to include both the object K and the morphism k.

Since you're now trained in category theory, you should immediately sense that there must also be a dual concept of a kernel as well: a **cokernel**. This is defined as the colimit to the diagram

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ X & \xrightarrow{f} & Y \end{array} \tag{3.43}$$

i.e. it is a morphism $c\colon Y \to C$ which is universal with respect to the property that $c \circ f = 0$. We'll denote the cokernel by $\mathrm{Coker}(f)$.

DEFINITION 3.2.2. A **preabelian** category is an additive category in which every morphism has a kernel and a cokernel.

EXAMPLE 3.2.3. The usual kernel as in (3.40) of a morphism in R-Mod is also a kernel in the categorical sense. In particular, any morphism has a kernel. The cokernel of a morphism $f\colon A \to B$ exists as well and is given by the quotient map

$$B \xrightarrow{=:\mathrm{Coker}(f)} B/\operatorname{Im}(f)\ . \tag{3.44}$$

In particular, R-Mod is preabelian.

EXAMPLE 3.2.4. The usual kernel as in (3.40) of a morphism in Grp is a kernel in the categorical sense. Cokernels are a bit more difficult. Let $f\colon G \to H$ be a morphism of groups. In contrast to abelian groups we can't take $H/\operatorname{Im}(f)$ as in (3.44) because the image will not necessarily be a normal subgroup and therefore we can't form the quotient. But we can apply a trick: we'll take the *normal closure* of $\operatorname{Im}(f)$, i.e. the smallest normal subgroup of H containing $\operatorname{Im}(f)$. Let's denote this by $\langle \operatorname{Im}(f)^H \rangle$. Then we can form

$$H \xrightarrow{=:\mathrm{Coker}(f)} H/\langle \operatorname{Im}(f)^H \rangle \tag{3.45}$$

and this is indeed a categorical cokernel. Hence, Grp has kernels and cokernels. But recall from Example 3.1.20 or Example 3.1.32 that Grp is not additive and thus not preabelian.

EXAMPLE 3.2.5. Let R be a ring. A finite direct sum of finitely generated R-modules is of course again finitely generated and it thus follows that the category R-mod of finitely generated R-modules is an R-linear additive category.

A quotient of a finitely generated module is finitely generated as well. Hence, the usual cokernel as defined in (3.44) for a morphism of finitely generated modules lives in R-mod and is clearly a cokernel there as well, i.e. R-mod has cokernels.

Things are more difficult for kernels. If all finitely generated R-modules have the property that all submodules are finitely generated as well, then the usual kernel as defined in (3.40) for a morphism of finitely generated modules lives in R-mod and is clearly a kernel there as well, i.e. R-mod has kernels. Rings with this property are called (left) **noetherian**.

Almost all rings you know are noetherian. Clearly, fields are noetherian. One can show that any ring which is a finite module over a noetherian ring is also noetherian. In particular, a finite-dimensional algebra over a field is noetherian. Moreover, there's **Hilbert's basis theorem** which says that any commutative ring which is finitely generated (as an algebra) over a commutative noetherian ring is itself noetherian.

But there are non-noetherian rings, e.g. the polynomial ring in infinitely many variables. If R is non-noetherian, then by definition there is a finitely generated R-module M having a non-finitely generated submodule U and since U is the kernel of the quotient map $q\colon M \to M/U$ and M/U is finitely generated, we have a morphism q in R-mod whose usual kernel is not contained in R-mod. Does this already mean that R-mod has no kernels at all? No, this is not yet clear! In principle R-mod could have kernels which look different from the kernels in R-Mod.[9] But in this particular case we can show that the inclusion $R\text{-mod} \to R\text{-Mod}$ preserves categorical kernels, so that a categorical kernel in R-mod really must be of the usual form (3.40). Hence, a morphism in R-mod has a kernel in R-mod if and only if its usual kernel is finitely generated. It thus follows that R-mod has kernels if and only if R is (left) noetherian.

It remains to show that a kernel in R-mod is also a kernel in its big brother R-Mod.[10] Let $f\colon M \to N$ be a morphism of finitely generated R-modules. Let $k\colon K \to M$ be a kernel in R-mod. The claim is that k is also a kernel in the bigger category R-Mod. So, let $k'\colon K' \to M$ be any morphism in R-Mod with $fk' = 0$. We need to show that k' factorizes uniquely through k, i.e. $k' = kt$ for a unique morphism $t\colon K' \to K$. By assumption, for any *finitely generated* submodule U of K' we have $fk'u = 0$, where $u\colon U \to K'$ is the inclusion. Hence, by the universal property of k in R-mod we have $k'u = kt_U$ for a unique morphism $t_U\colon U \to K$. Since any module is the union of its finitely generated submodules, it follows that a factorization $k' = kt$ is unique if it exists. But in turn using the uniqueness of the t_U, it follows that t_U and $t_{U'}$ agree on the intersection $U \cap U'$ and therefore the t_U glue to a morphism $t\colon K' \to K$ giving a factorization $k' = kt$.

Now, let's turn to submodules and quotients. What is a subgroup of an abelian group A? It's a subset $U \subseteq A$ which is closed under addition and taking negatives. Again, this is not a categorical formulation, so how can we categorize this? Well, associated to U is the inclusion $U \to A$. This is an injective morphism of abelian groups. Injectivity is still not a categorical concept but recall that we categorized this already in Section 1.3 by the notion of monomorphisms, and monomorphisms in Ab are the same as injective morphisms. Conversely, we can *view* a monomorphism $u\colon U \to A$ as a subgroup of A, namely as $\mathrm{Im}(u)$.

There's just one little issue that we tacitly ignored when we said above that we can *view* a monomorphism as a subobject. Namely, different monomorphisms can define the same subgroup: for example there are *two* monomorphisms $\mathbb{Z} \to \mathbb{Z}$ of abelian groups (one is the identity and the other one sends 1 to -1) but both define the subgroup $\mathbb{Z}$ of $\mathbb{Z}$. The classical notion of subgroups simply identifies all isomorphic subobjects. We can do the same by introducing an equivalence relation on monomorphisms.

Definition 3.2.6. Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. Given two monomorphisms $u\colon U \to X$ and $u'\colon U' \to X$ we write $u \leq u'$ if there is a morphism

[9] Pause a minute and think about this!

[10] The following nice argument is from `https://math.stackexchange.com/questions/1857330/why-is-the-category-of-finitely-generated-modules-over-a-non-noetherian-ring-not`.

$\varphi\colon U \to U'$ making the diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle u}\nearrow & & \nwarrow{\scriptstyle u'} \\ U & \xrightarrow[\varphi]{} & U' \end{array} \tag{3.46}$$

commutative.

Since u' in (3.46) is a monomorphism, there is at most one morphism φ making the diagram commutative. Moreover, φ is then also a monomorphism. The relation $\leq$ is a preorder on the class of all monomorphisms into X, i.e. it is reflexive and transitive. From this we get an equivalence relation $=$ on monomorphisms into X as follows:

$$u = u' :\Leftrightarrow u \leq u' \text{ and } u' \leq u\ . \tag{3.47}$$

Note that $u = u'$ if and only if there is a diagram as in (3.46) with φ being an *iso*morphism.

Definition 3.2.7. A **subobject** of X is an equivalence class of monomorphisms into X.

We write $\mathrm{Sub}_{\mathcal{C}}(X)$ for the class of subobjects of X. The preorder $\leq$ on monomorphisms descends to a partial order on $\mathrm{Sub}_{\mathcal{C}}(X)$.

Example 3.2.8. In R-Mod the subobjects of an R-module M are in bijection with the R-submodules of M and the partial order $\leq$ on subobjects is the same as the inclusion $\subseteq$ of submodules.

Dually, associated to a quotient A/U of abelian groups is the surjective morphism $A \to A/U$. Surjective morphisms of abelian groups are precisely the epimorphisms in Ab, and given an epimorphism $q\colon A \to Q$ we can *view* it as a quotient of A, namely as $A/\operatorname{Ker}(q)$. The same game works more generally for modules. We thus define a **quotient object** of X as an equivalence class of epimorphisms out of X.

Exercise 3.2.9. Let $f\colon X \to Y$ be a morphism in a category $\mathcal{C}$. Show that the kernel of f (if it exists) is a subobject of X, and that the cokernel of f (if it exists) is a quotient of Y.

When we discussed monomorphisms and epimorphisms in Section 1.3 we have seen quite strange behavior. For example, we have seen in Example 1.3.4 that the canonical ring morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in Ring, which by our new terminology means that $\mathbb{Z}$ is a quotient of $\mathbb{Q}$. So, you need to be a bit careful when applying your intuition to these general categorical concepts. But even in preabelian categories, subobjects and quotients do not yet have to behave like for modules—and this is the reason why these categories are only called *pre*abelian. Here's something that we would certainly want for a category behaving like R-Mod. Consider an abelian group A and a subgroup U, i.e. a monomorphism $u\colon U \to A$. The quotient of A by U is the epimorphism $q\colon A \to A/U$. Note that q is precisely the cokernel of u by (3.44). Now, the fact that we can form a quotient of A where we "precisely mod out U" is formalized by the two equations

$$\operatorname{Ker}(\operatorname{Coker}(u)) = u\ , \quad \operatorname{Coker}(\operatorname{Ker}(q)) = q\ . \tag{3.48}$$

There's a priori no reason why these equations should hold in a general category. In Example 3.2.4 we have seen that the category Grp has kernels and cokernels—great; but the cokernel of a monomorphism $u\colon U \to G$ is given by $q\colon G \to G/\langle U^G \rangle$, where $\langle U^G \rangle$ is the normal closure of U in G, and we thus have $\mathrm{Ker}(\mathrm{Coker}(q)) = U^G$, which is not necessarily equal to U. This observation is really just the fact that in Grp we cannot form the quotient by an arbitrary subgroup—we need *normal* subgroups. This is no problem in R-Mod, however, and we thus make the following definitions.

Definition 3.2.10. Let $\mathcal{C}$ be a category with kernels and cokernels.

(1) A monomorphism u in $\mathcal{C}$ is called **normal** if $\mathrm{Ker}(\mathrm{Coker}(u)) = u$.
(2) An epimorphism q in $\mathcal{C}$ is called **conormal** if $\mathrm{Coker}(\mathrm{Ker}(q)) = q$.

Exercise 3.2.11. Let $\mathcal{C}$ be a category with kernels and cokernels. Show that a monomorphism is normal if and only if it is the kernel of *some* morphism. Dually, show that an epimorphism is conormal if and only if it is the cokernel of *some* morphism. In the literature, you often see this apparently more general definition of (co)normality.

Definition 3.2.12. An **abelian** category is a preabelian category in which every monomorphism is normal and every epimorphism is conormal.

As for additivity, note that being abelian is a *property* of a category and not a structure.

Example 3.2.13. The category R-Mod is abelian.

Example 3.2.14. The category R-mod of finitely generated modules is abelian if and only if R is (left) noetherian. This follows from the discussion in Example 3.2.5. In particular, the category A-mod of finite-dimensional modules over a finite-dimensional algebra A over a field is abelian. As a special case, we obtain that the category $\mathsf{rep}_K(G)$ of finite-dimensional representations of a finite group G over a field K is abelian.

Example 3.2.15. If $\mathcal{C}$ is abelian, then the category $\mathsf{Gr}(\mathcal{C})$ of graded objects from Example 3.1.36 is abelian. The kernel and cokernel of a morphism are defined component-wise.

Example 3.2.16. Similarly, if $\mathcal{C}$ is abelian, then the categories $\mathsf{Ch}_\bullet(\mathcal{C})$ and $\mathsf{Ch}^\bullet(\mathcal{C})$ of (co)chain complexes over $\mathcal{C}$ from Example 3.1.37 are abelian.

Example 3.2.17. The category $\mathsf{Fun}(\mathcal{C}, \mathcal{C}')$ of functors from a category $\mathcal{C}$ into an abelian category $\mathcal{C}'$ is abelian.

Remark 3.2.18. There are preabelian categories which are not abelian, i.e. the normality conditions are violated (the example Grp which violates the normality conditions as well is not preabelian). An example is the category of torsion-free abelian groups.

Let $\mathcal{C}$ be an abelian category. If $u\colon U \to X$ is a subobject of X, then we define the **quotient** of X by U as

$$X/U \coloneqq \mathrm{Coker}(u) \; . \tag{3.49}$$

As usual, we drop the morphism u in the notation (and thinking) but keep in mind that it's always there. In R-Mod this categorical quotient is exactly the usual quotient of a module by a submodule. Question for you: in R-Mod, what is the kernel of the cokernel of a morphism f? Yes, it's precisely the image of f! So, let's define in general the **image** of a morphism f as

$$\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f)) . \tag{3.50}$$

We can now define the following fundamental concept. Consider a **sequence**

$$\cdots \xrightarrow{f_{i-2}} X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \xrightarrow{f_{i+1}} \cdots \tag{3.51}$$

of morphisms in an abelian category $\mathcal{C}$ indexed by integers i in some interval I. If $i \in I$ is such that also $i+1 \in I$, then the sequence is called **exact at position** i if

$$\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1}) , \tag{3.52}$$

and the sequence is called **exact** if it is exact at all positions $i \in I$ with $i+1 \in I$.

Exercise 3.2.19. Let $\mathcal{C}$ be an abelian category. Show that:

(1) a sequence $0 \to X \xrightarrow{f} Y$ is exact if and only if f is a monomorphism.
(2) a sequence $X \xrightarrow{f} Y \to 0$ is exact if and only if f is an epimorphism.
(3) a sequence $0 \to X \xrightarrow{f} Y \to 0$ is exact if and only if f is an isomorphism.

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow U \xrightarrow{u} X \xrightarrow{q} Q \longrightarrow 0 . \tag{3.53}$$

By Exercise 3.2.19, this means that u is a monomorphism (i.e. a subobject), q is an epimorphism (i.e. a quotient object), and $\mathrm{Im}(u) = \mathrm{Ker}(q)$. By definition of the image, the latter condition means that $\mathrm{Ker}(\mathrm{Coker}(u)) = \mathrm{Ker}(q)$. Now, taking the cokernel and using normality and conormality, we obtain

$$\mathrm{Coker}(u) = \mathrm{Coker}(\mathrm{Ker}(\mathrm{Coker}(u))) = \mathrm{Coker}(\mathrm{Ker}(q)) = q ,$$

i.e. in terms of objects we have

$$Q \simeq X/U . \tag{3.54}$$

Conversely, for any subobject $u\colon U \to X$ we obtain a short exact sequence as in (3.53) with $Q = X/U$. Hence, short exact sequences are a handy way to encode subobjects and their quotients.

Exercise 3.2.20. Show that an exact sequence can be split into a series of short exact sequences.

When we introduced additive categories, we also introduced their structure preserving functors, namely the additive functors. What are the structure preserving functors for abelian categories? Those preserving exact sequences!

Definition 3.2.21. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ between abelian categories is **exact** if it is additive and if for any short exact sequence

$$0 \longrightarrow U \xrightarrow{u} X \xrightarrow{q} Q \longrightarrow 0 \tag{3.55}$$

the induced sequence

$$0 \longrightarrow F(U) \xrightarrow{F(u)} F(X) \xrightarrow{F(q)} F(Q) \longrightarrow 0 \tag{3.56}$$

is exact.

An exact functor preserves kernels and cokernels. Moreover, by Exercise 3.2.20 an exact functor preserves exactness of sequences of arbitrary length.

Example 3.2.22. Let $\varphi\colon R \to S$ be a morphism of rings. Then any S-module W can naturally be viewed as an R-module W_R via

$$rw \coloneqq \varphi(r)w \tag{3.57}$$

for $r \in R$ and $w \in W$. An S-module morphism $f\colon W \to W'$ is automatically also an R-module morphism $f_R\colon W_R \to W'_R$. In total, we get a functor

$$(-)_R\colon S\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod} \tag{3.58}$$

called **scalar restriction**. This functor is obviously exact.

Exercise 3.2.23. Show that if $F\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories and $\mathcal{C}$ is abelian, then $\mathcal{C}'$ is abelian and F is exact.

As we did for additive categories in Definition 3.1.47, let's think about the appropriate notion of a *sub*category of an abelian category.

Definition 3.2.24. An **abelian subcategory** of an abelian category $\mathcal{C}'$ is an additive subcategory $\mathcal{C}$ of $\mathcal{C}'$ which is closed under kernels and cokernels in $\mathcal{C}'$, i.e. the kernel (cokernel) in $\mathcal{C}'$ of a morphism in $\mathcal{C}$ is also kernel (cokernel) in $\mathcal{C}$.

In this case, the category $\mathcal{C}$ is itself abelian and the natural functor $\mathcal{C} \to \mathcal{C}'$ is exact. As we noted for the direct sum in additive subcategories, if $\mathcal{C}$ is a *full* subcategory, then it is sufficient to check that the kernel (cokernel) *as an object* is contained in $\mathcal{C}$; but if $\mathcal{C}$ is not full, you need to check that the kernel (cokernel) in $\mathcal{C}'$ is really a kernel (cokernel) in $\mathcal{C}$ as well.

Example 3.2.25. If R is (left) noetherian, then R-mod is an abelian full subcategory of R-Mod.

There would be a lot to say about abelian categories—whole books have been written on them [5]. We cannot—and don't have to—delve into this subject here and finish with one fundamental result about abelian categories which can be very helpful when working with them. Abelian categories are defined in a way so that subobjects, quotients, exact sequences, and the like are defined and behave very similar to what we are used to from modules over a ring. Now, you should ask: do "basic" theorems like the **third isomorphism theorem**

$$(A/T)/(U/T) \simeq A/U \tag{3.59}$$

for an abelian group A and subgroups $T \subseteq U$ of A hold in any abelian category? And do we really need to reprove all this categorically now? One needs to be careful with the word "basic" but if we mean "general categorical statements about modules that can be phrased in terms of exact sequences"—like the isomorphism theorem (3.59)—then we don't have to reprove them: the reason is the amazing **Freyd–Mitchell embedding theorem** [5, 10].

Theorem 3.2.26. *If $\mathcal{C}$ is an* essentially small *abelian category, then there is a ring R and an exact embedding $F\colon \mathcal{C} \to R$-$\mathsf{Mod}$, i.e. $\mathcal{C}$ is equivalent to an abelian full subcategory of R-Mod.*

The proof needs some considerable amount of work and we will just take this as a fact here. Before I'll show how to use the theorem in practice, I want to make some remarks.

Remark 3.2.27. The basic idea of the proof of Theorem 3.2.26 is to embed $\mathcal{C}$ into the full subcategory $\mathcal{L}$ of $\mathsf{Fun}(\mathcal{C}, \mathsf{Ab})$ consisting of so-called left exact functors, and then show that $\mathcal{L}$ is equivalent to R-Mod, where $R \coloneqq \mathrm{End}_{\mathcal{L}}(I)$ for a special object I of $\mathcal{L}$. The ring R is not explicit and therefore it's still absolutely justified developing general theory of abelian categories.

In many steps of the proof—e.g. for showing that $\mathsf{Fun}(\mathcal{C}, \mathsf{Ab})$ is a so-called Grothendieck category (which has nice properties) and that the special object I exists—you need that $\mathcal{C}$ is a *small* category—this is why the size assumption is in the theorem. Once you have proven the theorem for small categories, you get it quickly extended to *essentially* small categories because if $\mathcal{C}$ is an essentially small category, then by definition there is an equivalence $F \colon \mathcal{C} \to \mathcal{C}'$ to a small category $\mathcal{C}'$; the category $\mathcal{C}'$ is abelian and F is exact by Exercise 3.2.23, hence by the theorem you get an exact embedding $\mathcal{C}' \to R$-Mod; composed with F you get an exact embedding $\mathcal{C} \to R$-Mod.

You could ask whether the size assumption is simply because we don't have a better proof. But it's evident that there needs to be some sort of size restriction because there are some size restrictions already in R-Mod. Namely, since F is an embedding, it induces an injection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{R\text{-}\mathsf{Mod}}(F(X), F(Y)) \tag{3.60}$$

for all $X, Y \in \mathcal{C}$. Since R-Mod is locally small, it follows that $\mathcal{C}$ must be locally small as well. Moreover, since F is exact, it induces for any object $X \in \mathcal{C}$ an injection

$$\mathrm{Sub}_{\mathcal{C}}(X) \to \mathrm{Sub}_{R\text{-}\mathsf{Mod}}(F(X)) \; . \tag{3.61}$$

In particular, the class $\mathrm{Sub}_{\mathcal{C}}(X)$ needs to be a *set* because the collection of submodules of a module is a set! Categories for which $\mathrm{Sub}_{\mathcal{C}}(X)$ is a set for all objects X are called **well-powered**. Small categories are certainly well-powered, and a category equivalent to a well-powered category is well-powered. Hence, essentially small categories are well-powered. But there are (locally small) abelian categories which are not well-powered[11] and which therefore cannot be embedded into R-Mod.[12]

Now, let's use Theorem 3.2.26 to prove the third isomorphism theorem (3.59) in any abelian category $\mathcal{C}$. Let $X \in \mathcal{C}$ and let $T \leq U \leq X$ be subobjects. If $\mathcal{C}$ is essentially small, then we get an exact embedding $F \colon \mathcal{C} \to R$-$\mathsf{Mod}$. Because F is exact, it maps subobjects to subobjects and commutes with taking quotients. Hence, by using the isomorphism theorem for modules we get

$$F((X/T)/(U/T)) \simeq F(X/T)/F(U/T) \simeq (F(X)/F(T)) \,/\, (F(U)/F(T)) \tag{3.62}$$

$$\simeq F(X)/F(U) \simeq F(X/U) \; . \tag{3.63}$$

Since F is an embedding and thus an equivalence onto its full image, it follows that

$$(X/T)/(U/T) \simeq X/U \; . \tag{3.64}$$

In a similar fashion you can prove that any "general categorical statement about modules that can be phrased in terms of exact sequences" also holds in any abelian

[11] See e.g. `https://mathoverflow.net/questions/93853/abelian-category-which-is-not-well-powered`.

[12] I assume that locally small and well-powered also won't be enough to embed into R-Mod but I don't know a counter-example right now. My point was only to convince you that you certainly need some size restrictions.

category—you just need to make sure that you only deal with statements that are preserved by an exact embedding.

There's only one minor issue left. To apply Theorem 3.2.26 we need to assume that $\mathcal{C}$ is essentially small. What if $\mathcal{C}$ is not essentially small? The third isomorphism theorem (3.64) just deals with three objects. If we could fit them into a small abelian full subcategory, then we could deduce the third isomorphism theorem for any abelian category from the one we have just proven for small abelian categories. This is indeed always possible:

LEMMA 3.2.28. *Let $\mathcal{C}$ be an abelian category. Then any* set *$\mathcal{X}$ of objects in $\mathcal{C}$ lies in a small abelian full subcategory of $\mathcal{C}$.*

PROOF. We inductively construct a sequence $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of small full subcategories of $\mathcal{C}$ as follows. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ whose set of objects is equal to $\mathcal{X}$. Now, let $i > 0$ and assume $\mathcal{C}_i$ is already constructed. We then let $\mathcal{C}_{i+1}$ be the full subcategory of $\mathcal{C}$ consisting of a choice of kernel and cokernel of every morphism in $\mathcal{C}_i$ and of a choice of direct sum for any finite set of objects in $\mathcal{C}_i$. Here, the kernel, cokernel, and direct sum are those in $\mathcal{C}$. Since $\mathcal{C}_i$ is small, the category $\mathcal{C}_{i+1}$ is small as well. Moreover, we have $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$. Hence, $\mathcal{C}' := \bigcup_{i\in\mathbb{N}} \mathcal{C}_i$ is a small full subcategory of $\mathcal{C}$ containing $\mathcal{X}$. Since $\mathcal{C}'$ is a full subcategory and closed under finite direct sums, it is an additive subcategory. Moreover, by construction $\mathcal{C}'$ is closed under taking kernels and cokernels. Hence, $\mathcal{C}'$ is an abelian subcategory. □

EXERCISE 3.2.29. Prove the **nine lemma** in an abelian category $\mathcal{C}$: if

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & Z_1 & & Z_2 & & Z_3 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.65}$$

is a commutative diagram with exact rows and exact columns, then there are uniquely determined morphisms $Z_1 \to Z_2$ and $Z_2 \to Z_3$ making the diagram commutative. Moreover, the sequence $0 \to Z_1 \to Z_2 \to Z_3 \to 0$ is exact. Use this lemma to (re)prove the third isomorphism theorem.

We want to say a bit more about subobjects in an abelian category. We have categorized the notion of subgroups of an abelian group via (equivalence classes of) subobjects and the inclusion of subgroups via the relation $\leq$. There are a few more things we can do with subgroups. If A is an abelian group und U_1, U_2 are subgroups, we can form their intersection $U_1 \cap U_2$ and we can take their "union" $U_1 \cup U_2$ by which I actually mean their sum $U_1 + U_2$ but I want to use the symbol $\cup$ to avoid excessive use of the symbol $+$. The intersection and union have an order-theoretic meaning: they are the **infimum** (greatest lower bound) and **supremum** (least upper bound), respectively, of U_1 and U_2 in the partially ordered set of subgroups of A. From this point of view the $\cup$ symbol is also justified. A partially ordered set in which every two elements have an infimum and a supremum is called a **lattice**. Note that infimum and supremum are unique if they exist.

We want to categorize intersections and sums for subobjects in an abelian category $\mathcal{C}$. Let's start with the union. Let $X \in \mathcal{C}$ and let $u_i\colon U_i \to X$ for $i = 1, 2$ be two subobjects of X. The universal property of the direct sum $U_1 \oplus U_2$ applied to u_i yields a unique morphism

$$u_{12}\colon U_1 \oplus U_2 \to X \ , \tag{3.66}$$

with the property that $u_i = u_{12} \circ \mathrm{i}_i$, where $\mathrm{i}_i\colon U_i \to U_1 \oplus U_2$ is the inclusion. In case of abelian groups, this is the map $(a_1, a_2) \mapsto a_1 + a_2$ whose image is precisely $U_1 \cup U_2$, which is what we want. So, let's just take

$$u_1 \cup u_2 \coloneqq \mathrm{Im}(u_{12}) = \mathrm{Ker}(\mathrm{Coker}(u_{12})) \ . \tag{3.67}$$

Let's write $U_1 \cup U_2$ for the domain of $u_1 \cup u_2$. Since kernels are monomorphisms, it follows that $u_1 \cup u_2$ is a subobject of X. We did the right thing because:

Lemma 3.2.30. *$u_1 \cup u_2$ is the supremum of u_1 and u_2 in $\mathrm{Sub}_{\mathcal{C}}(X)$.*

Proof. We need to show that $u_i \leq u_1 \cup u_2$ and that if $u\colon U \to X$ is any subobject of X with $u_i \leq u$, then $u_1 \cup u_2 \leq u$. One could prove this by general nonsense but I have a simpler suggestion: let's use the Freyd–Mitchell embedding theorem (Theorem 3.2.26)! For modules, the claim is obvious. By Lemma 3.2.28 we can find a small abelian full subcategory $\mathcal{C}'$ of $\mathcal{C}$ containing all the objects in question, i.e. containing U_i, U, and X. Let $F\colon \mathcal{C}' \to R\text{-}\mathsf{Mod}$ be an exact embedding into a module category. Then by construction $F(u_1 \cup u_2) = F(u_1) \cup F(u_2)$. Now, $F(u_i) \leq F(u_1) \cup F(u_2)$ is clear and it follows that also $u_i \leq u_1 \cup u_2$. Moreover, $F(u) \leq F(u_1) \cup F(u_2)$, hence $u \leq u_1 \cup u_2$.[13] □

Let's come to the intersection. Let $q_i \coloneqq \mathrm{Coker}(u_i)\colon X \to X/U_i \eqqcolon Q_i$ be the quotient morphism. The universal property of the direct sum $Q_1 \oplus Q_2$ applied to q_i yields a unique morphism

$$q_{12}\colon X \to Q_1 \oplus Q_2 \tag{3.68}$$

with the property that $\mathrm{p}_i \circ q_{12} = q_i$, where $\mathrm{p}_i\colon Q_1 \oplus Q_2 \to Q_i$ is the projection. In case of abelian grops, the kernel of q_{12} is precisely the intersection $U_1 \cap U_2$. So, let's just take

$$u_1 \cap u_2 \coloneqq \mathrm{Ker}(q_{12}) \ . \tag{3.69}$$

Let's write $U_1 \cap U_2$ for the domain of $u_1 \cap u_2$. Again, since kernels are monomorphisms, it follows that $u_1 \cap u_2$ is a subobject of X. Similarly as for the union you prove that $u_1 \cap u_2$ is the infimum of u_1 and u_2 in $\mathrm{Sub}_{\mathcal{C}}(X)$.

We have thus shown that for any object X of an abelian category $\mathcal{C}$, any two elements in $\mathrm{Sub}_{\mathcal{C}}(X)$ have an infimum and a supremum. Hence, if $\mathrm{Sub}_{\mathcal{C}}(X)$ is actually a set, i.e. if $\mathcal{C}$ is well-powered (e.g. if $\mathcal{C}$ is essentially small), then $\mathrm{Sub}_{\mathcal{C}}(X)$ is a *lattice*. Nice!

[13]Note that we could get another embedding for varying u but this does not matter since all we want to conclude is that $u_i \leq u_1 \cup u_2$ and $u \leq u_1 \cup u_2$ for any given u.

Exercise 3.2.31. Show that the **second isomorphism theorem** holds in any abelian category $\mathcal{C}$: given subobjects U_1 and U_2 of an object $X \in \mathcal{C}$, then

$$(U_1 \cup U_2)/U_2 \simeq U_1/(U_1 \cap U_2) \ . \tag{3.70}$$

Hint: use the nine lemma (Exercise 3.2.29).

Remark 3.2.32. The study of exactness (and non-exactness) of sequences and functors is called **homological algebra**. This topic may sound harmless and boring but it is a fundamental part of mathematics—whole books have been written on this subject as well. Abelian categories are the natural general stage to do homological algebra.[14] To get a minimal taste of homological algebra, recall the example of chain complexes from Example 3.1.37. In the definition of a complex $A^\bullet = (A_i, d_i)$ we had the strange condition $d_{i-1} \circ d_i = 0$ on the differentials. For complexes over an abelian category, you can quickly check that this is equivalent to the property that

$$\mathrm{Im}(d_i) \subseteq \mathrm{Ker}(d_{i-1}) \tag{3.71}$$

and therefore we can define for each i the i-th **homology object**

$$\mathrm{H}_i(A_\bullet) \coloneqq \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i+1}) \ . \tag{3.72}$$

The complex $A^\bullet$ is exact if and only if all homology objects are zero. Hence, the homology measures "how far" the complex is from being exact! Similarly, for a cochain complex $A^\bullet$ we can define the **cohomology object**

$$\mathrm{H}^i(A^\bullet) \coloneqq \mathrm{Ker}(d^i)/\,\mathrm{Im}(d^{i-1}) \ . \tag{3.73}$$

Homological algebra is concerned with problems like finding tools to compute the (co)homology etc. The motivation for all this is that there are (surprisingly) many situations in practice where you can define on a category $\mathcal{C}$ a functor $C \colon \mathcal{C} \to \mathsf{Ch}_\bullet(R)$ to chain complexes over a ring R, say, and since C is a functor, the homology objects $\mathrm{H}_i(X) \coloneqq \mathrm{H}_i(C_\bullet(X))$ of the chain complex $C_\bullet(X)$ associated to $X \in \mathcal{C}$ are (algebraic) *invariants* of X. For example, one can associate to any topological space X the so-called **singular chain complex** $C_\bullet(X) \in \mathsf{Ch}_\bullet(\mathbb{Z})$. The i-th term of this complex consists of the free abelian group on the continuous images of i-simplices in X. The homology groups $\mathrm{H}_i(X) \coloneqq \mathrm{H}_i(C_\bullet(X))$ are called **singular homology** groups of X. They are very important topological invariants. Intuitively, $\mathrm{H}_i(X)$ counts the i-dimensional holes in X.

3.3. Finite categories

So far, all categories were quite general. This is nice because we can cover a lot of examples but it's also hard because a general category can be arbitrarily complicated and we don't really have a starting point for studying it. We'll now come to a special class of categories which has such a starting point and arises frequently in representation theory (and in the theory of tensor categories). All objects in these categories are "built up" from "simple" objects. So, you would start with describing the simple objects and then see how far you can get—but at least there's a starting point.

Throughout, $\mathcal{C}$ denotes an abelian category.

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[14] There's a more general notion of **exact categories** for which a lot of homological algebra still works as usual.

DEFINITION 3.3.1. An object $X \in \mathcal{C}$ is **simple** if it is not (isomorphic to) the zero object and has no subobjects other than 0 and X.

EXAMPLE 3.3.2. In K-Vec the only simple object (up to isomorphism) is the one-dimensional vector space K.

In the example K-Vec, it's already clear that any object is "built up" from simple objects in the sense that any vector space is a direct sum of copies of the unique simple object K. We'll be mostly concerned with categories where objects are "built up" from *finitely* many simple objects. This is true for example for the finite version K-vec. On the other hand, it's quite a strong condition already that objects are *direct sums* of simple objects—such categories are called **semisimple** and will be discussed in more detail in Section 3.4. I first want to focus on a weaker property.

DEFINITION 3.3.3. A **composition series** of an object $X \in \mathcal{C}$ is a descending finite chain

$$0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = X \tag{3.74}$$

of subobjects of X such that each successive quotient X_i/X_{i-1} is simple. The integer n is called the **length** of the composition series.

We want to prove the **Jordan–Hölder theorem**:

THEOREM 3.3.4. *Suppose that X has a composition series. Then any two composition series of X are **equivalent** in the sense that if*

$$0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = X$$

and

$$0 = X'_0 < X'_1 < \cdots < X'_{m-1} < X'_m = X$$

are two composition series of X, then $m = n$ and there is a permutation σ on the indices such that

$$X_i/X_{i-1} \simeq X'_{\sigma(i)}/X'_{\sigma(i)-1} \tag{3.75}$$

*for all i. In particular, the length of a composition series and the **multiplicity***

$$[X : S] := \#\{i \mid X_i/X_{i-1} \simeq S\} \tag{3.76}$$

of a simple object S as simple quotient in a composition series of X are independent of the composition series.

For the proof, we'll need two lemmas. But first, I want to make a remark.

REMARK 3.3.5. Maybe you know the Jordan–Hölder theorem already for modules and wonder why we don't simply use the Freyd–Mitchell embedding theorem to quickly deduce the Jordan–Hölder theorem for arbitrary abelian categories. The problem is that being simple is not a property in terms of exact sequences and is therefore not necessarily preserved by an exact embedding: if $F : \mathcal{C} \to R$-Mod is an exact embedding and X is simple, then certainly $F(X)$ is a simple object in its full image but there is no reason why $F(X)$ should be a *simple module*, i.e. a simple object in the big category R-Mod; there may well be a submodule U of $F(X)$ which is not contained in the image of F. So, not everything is solved by using the Freyd–Mitchell embedding theorem. Nonetheless, the proof of the Jordan–Hölder theorem works exactly like the one for modules.

Lemma 3.3.6. *If $U_1 < X$ and $U_2 < X$ are two distinct subobjects of X such that X/U_i is simple, then $X = U_1 \cup U_2$.*

Proof. We have $U_i \leq U_1 \cup U_2$. Since $U_1 \neq U_2$, we must have $U_i \neq U_1 \cup U_2$ for some i. Without loss of generality, we can assume that $i = 2$. From the nine lemma Exercise 3.2.29 we get a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_2 & \longrightarrow & U_2 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_1 \cup U_2 & \longrightarrow & X & \longrightarrow & X/(U_1 \cup U_2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (U_1 \cup U_2)/U_2 & \longrightarrow & X/U_2 & \longrightarrow & X/(U_1 \cup U_2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{3.77}$$

with exact rows and exact columns. From the bottom row we get a monomorphism $(U_1 \cup U_2)/U_2 \to X/U_2$, i.e. $(U_1 \cup U_2)/U_2$ is a subobject of X/U_2. Since $(U_1 \cup U_2)/U_2 \neq 0$ and X/U_2 is simple by assumption, we must have $U_1 \cup U_2 = X$. □

Lemma 3.3.7. *Suppose that X has a composition series*

$$0 = X_0 < X_1 < \cdots < X_n = X \ .$$

Let $U < X$ with X/U simple. Then there is a composition series of the form

$$0 = X'_0 < \cdots < X'_{n-2} < U < X \ . \tag{3.78}$$

In particular, U has a composition series as well.

Proof. We'll prove this by induction on n. The case $n = 0$ is clear and if $n = 1$, then X is already simple and the claim is clear as well. Now, assume that $n > 1$. If $U = X_{n-1}$, the claim is clear, so we assume $U \neq X_{n-1}$. Then it follows from Lemma 3.3.6 that $X = U \cup X_{n-1}$. The second isomorphism theorem (Exercise 3.2.31) yields

$$X/U = (U \cup X_{n-1})/U \simeq X_{n-1}/(X_{n-1} \cap U) \ ,$$

which implies that $X_{n-1}/(X_{n-1} \cap U)$ is simple. The object X_{n-1} has a composition series of length $n - 1$. Hence, by induction there is a composition series

$$0 = X'_0 < X'_1 < \cdots < X'_{n-3} < X'_{n-2} \coloneqq X_{n-1} \cap U < X'_{n-1} = X_{n-1} \ .$$

In particular, $X_{n-1} \cap U$ has a composition series of length $n - 2$. This can be completed to a composition series

$$0 = X'_0 < X'_1 < \cdots < X'_{n-3} < X'_{n-2} < U < X$$

since

$$U/X'_{n-2} = U/(U \cap X_{n-1}) \simeq (U \cup X_{n-1})/X_{n-1} = X/X_{n-1}$$

is simple by the second isomorphism theorem. □

Proof of Theorem 3.3.4. The proof is by induction on the minimal length n of a composition series of X. The claim is clear for $n = 0, 1$ since there is only one composition series in this case. Now, let $n > 1$. Consider two composition series of X as in the statement of Theorem 3.3.4. We can assume that $X_{n-1} \neq X'_{m-1}$ since otherwise the statement can be reduced to X_{n-1}. Then by Lemma 3.3.6 we have $X = X_{n-1} \cup X'_{m-1}$. By the second isomorphism theorem we have

$$X/X_{n-1} = (X_{n-1} \cup X'_{m-1}) \simeq X'_{m-1}/(X_{n-1} \cap X'_{m-1}) \; ,$$

which is simple. Hence, by Lemma 3.3.7 there is a composition series of X_{n-1} of the form

$$0 = X''_0 < X''_1 < \cdots < X''_{n-3} < X_{n-1} \cap X'_{m-1} < X_{n-1} \; .$$

But also

$$X/X'_{m-1} = (X_{n-1} \cup X'_{m-1})/X'_{m-1} \simeq X_{n-1}/(X_{n-1} \cap X'_{m-1}) < X_{n-1}$$

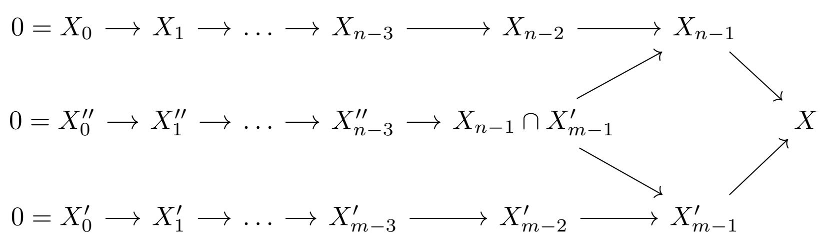
is simple and therefore

$$0 = X''_0 < X''_1 < \cdots < X''_{n-3} < X_{n-1} \cap X'_{m-1} < X'_{m-1}$$

is a composition series of X'_{m-1} of length $n - 1$. In particular, the minimal length of composition series of X'_{m-1} is $\leq n - 1$. Since

$$0 = X'_0 < X'_1 < \ldots < X'_{m-2} < X'_{m-1}$$

is another composition series of X'_{m-1}, it follows by induction that $m - 1 = n - 1$, i.e. $m = n$. We're now in the following situation:



From this diagram it follows by induction that the simple quotients (with their multiplicity) of a composition series of X are independent of the composition series. □

Definition 3.3.8. An object $X \in \mathcal{C}$ admitting a composition series is said to be of **finite length**. The length of one (any) composition series is called the **length** of X and is denoted by $\ell(X)$. A **length category** is an essentially small abelian category in which all objects are of finite length.

Example 3.3.9. The category K-vec is a length category and $\ell(V) = \dim_K(V)$. More generally, it is a classical fact that a finitely generated module over an **artinian** ring R (a ring satisfying the descending chain condition on ideals) has a composition series.[15] Hence, R-mod for an artinian ring R is a length category. In particular, A-mod for a finite-dimensional algebra A over a field K is a length category. As a special case, we deduce that $\mathsf{rep}_K(G)$ for a finite group G is a length category.

[15] You can find a proof in my notes `https://ulthiel.com/math/wp-content/uploads/commutative-algebra-2016-2017-stuttgart/Vorlesung-20.pdf`.

Module categories of finite-dimensional algebras over fields give the most important examples of length categories and we will now introduce some terminology to capture their additional structure.

DEFINITION 3.3.10. A **locally finite** category over a field K is a K-linear length category $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finite-dimensional K-vector space for any $X, Y \in \mathcal{C}$. A **finite** category over K is a category $\mathcal{C}$ which is equivalent to A-mod for a finite-dimensional K-algebra A.

Note that a finite category over K is K-linear (by transport of structure) and locally finite. One can give an internal characterization of finite categories, see [4] for more details.

LEMMA 3.3.11 (**Schur's lemma**). *Let $\mathcal{C}$ be an abelian category.*

(1) *If $S, T \in \mathcal{C}$ are non-isomorphic simple objects, then $\mathrm{Hom}_{\mathcal{C}}(S,T) = 0$.*
(2) *If $S \in \mathcal{C}$ is simple, then $\mathrm{End}_{\mathcal{C}}(S)$ is a division ring, i.e. every non-zero element is invertible.*

PROOF. Assume that S and T are simple objects. Let $f\colon S \to T$ be a non-zero morphism. Note that $\mathrm{Im}(f)$ is a subobject of T. Since f is non-zero and T is simple, we must have $\mathrm{Im}(f) = T$. Similarly, it follows that $\mathrm{Ker}(f) = 0$ and therefore f is already an isomorphism. Hence, if S and T are non-isomorphic, then $\mathrm{Hom}_{\mathcal{C}}(S,T) = 0$. Moreover, $\mathrm{End}_{\mathcal{C}}(S)$ is a division ring. □

COROLLARY 3.3.12. *If $\mathcal{C}$ is a locally finite K-linear abelian category over an algebraically closed field K, then we have $\mathrm{End}_{\mathcal{C}}(S) = K$ for any simple object $S \in \mathcal{C}$.*

PROOF. By Lemma 3.3.11 we know that $\mathrm{End}_{\mathcal{C}}(S)$ is a division algebra over K, and the locally finiteness assumption implies that this algebra is of finite-dimension over K. Since K is algebraically closed, there are no finite-dimensional division algebras over K except K itself. Maybe you don't know this fact, so let's prove it. Let D be a finite-dimensional division algebra over K. Let $x \in D$. Then the powers of x are linearly dependent over K since D is finite-dimensional. Hence, there is a polynomial

$$f = X^n + c_1 X^{n-1} + \ldots + c_n \in K[X] \tag{3.79}$$

with $f(x) = 0$. We assume that f is of minimal degree with this property. Since K is algebraically closed, f has a zero λ in K and we can write $f = (X - \lambda) \cdot g$ for $g \in K[X]$ of degree $n-1$. We have $0 = f(x) = (x-\lambda)\cdot g(x)$ and therefore $x - \lambda = 0$ or $g(x) = 0$. But f was of minimal degree with the property that $f(x) = 0$. Hence, $g(x) \neq 0$ and we must have $x - \lambda = 0$, i.e. $x = \lambda \in K$. This shows that $D = K$. □

3.4. Semisimple categories

Let's come back to the special case where objects do not just have a composition series but even are a direct sum of simple objects.

DEFINITION 3.4.1. Let $\mathcal{C}$ be an abelian category. An object $X \in \mathcal{C}$ is **semisimple** if it is isomorphic to a finite direct sum of simple objects, i.e.

$$X \simeq \bigoplus_{S \in \mathcal{S}} S\ , \tag{3.80}$$

where $\mathcal{S}$ is a finite set of simple objects in $\mathcal{C}$. A **semisimple** category is an abelian category in which every object is semisimple.

EXAMPLE 3.4.2. The category K-vec is semisimple.

We would really like the direct sum decomposition of a semisimple object to be unique up to permutation of the summands. This is not clear a priori. But from a decomposition as in (3.80) we obtain a composition series of X as follows. Choose a numbering $\mathcal{S} = \{S_1, \ldots, S_n\}$ of the simple objects in $\mathcal{S}$. For each $i = 1, \ldots, n$ define

$$X_i \coloneqq \bigoplus_{j=1}^{i} S_j \ . \tag{3.81}$$

Then

$$0 = X_0 < X_1 < \ldots < X_n = X \tag{3.82}$$

is a composition series with quotients

$$X_i / X_{i-1} \simeq S_i \ . \tag{3.83}$$

The Jordan–Hölder theorem now implies that the S_i are unique up to permutation, and consequently the direct sum decomposition of a semisimple object is unique up to permutation of the summands.

What are examples of semisimple categories away from the trivial example K-vec? Here's a fantastic classical theorem.

THEOREM 3.4.3 (**Maschke's theorem**). *Let K be a field and let G be a finite group. The category $\mathsf{rep}_K(G)$ is semisimple if and only if the characteristic of K does not divide the order of G.*

PROOF. Assume that the characteristic of K does not divide the order of G. Let $\rho \colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. We will show that any subrepresentation U of V has a **complement** as a representation, i.e. there is a subrepresentation U' of V such that $V = U \oplus U'$. Since V is finite-dimensional, we can then inductively decompose V into a direct sum of simple representations.

To prove the claim, let U'' be a complement of U in V as a vector space (this certainly exists). Let $p \colon V \to V$ be the linear map which is the identity on U and which is zero on U''. Define a new map $p' \colon V \to V$ by

$$p' \coloneqq \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho(g^{-1})$$

and let

$$U' \coloneqq \mathrm{Ker}(p') \ .$$

Note that p' is well-defined since by assumption $|G|$ is a non-zero element in K and therefore we can form $\frac{1}{|G|}$. We claim that U' is a complement of U in V as a representation. Since U is a subrepresentation, it is stable under all the $\rho(g)$, and since p is the identity on U, it follows that p' is the identity on U as well. Moreover, p' maps V into U. Hence, $(p')^2 = p'$, i.e. p' is a projection and therefore $V = U \oplus U'$ as vector spaces. It remains to show that U' is stable under the $\rho(g)$. For $u' \in U'$

we have

$$\begin{aligned} p' \circ \rho(g)(u') &= \frac{1}{|G|} \sum_{h \in G} \rho(h) \circ p \circ \rho(h^{-1}) \circ \rho(g)(u') \\ &= \frac{1}{|G|} \sum_{h \in G} \rho(h) \circ p \circ \rho(h^{-1} g)(u') \\ &= \frac{1}{|G|} \sum_{h \in G} \rho(gh) \circ p \circ \rho(h^{-1})(u') \\ &= \rho(g) \circ \frac{1}{|G|} \sum_{h \in G} \rho(h) \circ p \circ \rho(h^{-1})(u') \\ &= \rho(g) \circ p'(u') = 0 \ , \end{aligned}$$

i.e. $\rho(g)(u') \in U'$.

We still need to prove the converse of the statement in the theorem, i.e. if the characteristic of K divides the order of G, then $\mathsf{rep}_K(G)$ is not semisimple. It will be more convenient to work with KG-modules here. We define a map

$$\begin{array}{rcl} \varepsilon\colon KG & \to & K \\ \sum_{g \in G} \alpha_g g & \mapsto & \sum_{g \in G} \alpha_g \ . \end{array} \tag{3.84}$$

The map ε is a K-algebra morphism and therefore the kernel I of ε is a KG-submodule of KG. We will show that $I \cap U \neq \emptyset$ for any proper submodule U of KG. This implies that I does not have a complement in KG and therefore the KG-module KG is not semisimple. So, let $u = \sum_{g \in G} \alpha_g g$ be a non-zero element of U. If $\varepsilon(u) = 0$, then $u \in I$ by definition and so the claim holds. Otherwise, let $\gamma \coloneqq \sum_{g \in G} g \in KG$. Since U is a KG-submodule, we have $\gamma u \in U$. Moreover,

$$\gamma u = \left(\sum_{g \in G} g \right) \left(\sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \left(\sum_{h \in G} \alpha_h h \right) g = \varepsilon(u) \gamma \ .$$

This shows on the one hand that $\gamma u \neq 0$ and on the other hand

$$\varepsilon(\gamma u) = \varepsilon(\varepsilon(u)\gamma) = \varepsilon(u)|G| = 0 \ ,$$

i.e. $0 \neq \gamma u \in I \cap U$. □

3.5. Grothendieck groups

Throughout this section, we assume that $\mathcal{C}$ is an essentially small category. The collection $[\mathcal{C}]$ of isomorphism classes of $\mathcal{C}$ thus forms a *set*.

If $\mathcal{C}$ is (semi)additive, then we know from Exercise 3.1.22 that $[\mathcal{C}]$ becomes a commutative monoid with respect to the addition

$$[X] + [Y] \coloneqq [X \oplus Y] \ . \tag{3.85}$$

By Example 2.5.8 we can turn any commutative monoid M into an abelian group by taking the Grothendieck group $G(M)$. The abelian group

$$[\mathcal{C}]_{\oplus} \coloneqq G(([\mathcal{C}], +)) \tag{3.86}$$

is called the **split Grothendieck group** of $\mathcal{C}$. Alternatively, $[\mathcal{C}]_{\oplus}$ can be described as the quotient of the free abelian group (i.e. free $\mathbb{Z}$-module) $\mathbb{Z}^{[\mathcal{C}]}$ with basis $[\mathcal{C}]$ by the subgroup generated by the relation (3.85)

$$[X] - [X \oplus Y] + [Y] = 0\,. \tag{3.87}$$

If $\mathcal{C}$ is abelian, there's another point of view on the split Grothendieck group which also explains the prefix "split". First, note that for any two objects X and Y of $\mathcal{C}$ we have a canonical short exact sequence

$$0 \longrightarrow X \xrightarrow{\ \mathrm{i}\ } X \oplus Y \xrightarrow{\ \mathrm{p}\ } Y \longrightarrow 0\,, \tag{3.88}$$

where $\mathrm{i}\colon X \to X \oplus Y$ is the inclusion and $\mathrm{p}\colon X \oplus Y \to Y$ is the projection. We call a short exact sequence

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0 \tag{3.89}$$

split if it is isomorphic to the short exact sequence (3.88), i.e. there is an isomorphism $f\colon Z \to X \oplus Y$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \simeq\big\downarrow f & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{\ \mathrm{i}\ } & X \oplus Y & \xrightarrow{\ \mathrm{p}\ } & Y & \longrightarrow & 0 \end{array} \tag{3.90}$$

commutative. Now, it's clear that the split Grothendieck group $[\mathcal{C}]_\oplus$ is isomorphic to the quotient $\mathbb{Z}^{[\mathcal{C}]}$ by the subgroup generated by the relation

$$[X] - [Z] + [Y] = 0 \tag{3.91}$$

whenever there is a split short exact sequence like (3.89).

Instead of considering only *split* short exact sequences, it makes sense to consider *all* short exact sequences as well. The quotient of $\mathbb{Z}^{[\mathcal{C}]}$ by the subgroup generated by the relation

$$[X] - [Z] + [Y] = 0 \tag{3.92}$$

whenever there is a (not necessarily split) short exact sequence

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0 \tag{3.93}$$

is called the (abelian) **Grothendieck group** of $\mathcal{C}$ and is here denoted by $[\mathcal{C}]_{\mathrm{ab}}$. Clearly, $[\mathcal{C}]_{\mathrm{ab}}$ is a quotient of $[\mathcal{C}]_\oplus$.

We know from Example 2.5.8 that the split Grothendieck group $[\mathcal{C}]_\oplus$ satisfies a universal property, namely the canonical map $[\mathcal{C}] \to [\mathcal{C}]_\oplus$ is the universal monoid morphism into an abelian group in the sense that if $f\colon [\mathcal{C}] \to G$ is any other monoid morphism into an abelian group G, then it uniquely factors through $[\mathcal{C}] \to [\mathcal{C}]_\oplus$, i.e. there is a commutative diagram

$$\begin{array}{ccc} [\mathcal{C}]_\oplus & \overset{\exists!\tilde{f}}{\dashrightarrow} & G \\ \uparrow & \nearrow_{f} & \\ [\mathcal{C}] & & \end{array} \tag{3.94}$$

Does the Grothendieck group $[\mathcal{C}]_{\mathrm{ab}}$ satisfy a universal property as well? Yes: it's universal for **additive functions** on $\mathcal{C}$. These are maps

$$\chi\colon \mathcal{C} \to G \tag{3.95}$$

into an abelian group G such that

$$\chi(X) - \chi(Z) + \chi(Y) = 0 \tag{3.96}$$

whenever there is a short exact sequence

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0\,. \tag{3.97}$$

The natural map $\mathcal{C} \to [\mathcal{C}]_{\mathrm{ab}}$, $X \mapsto [X]$, is additive, and it is the universal additive function since any other additive function $\chi\colon \mathcal{C} \to G$ clearly uniquely factors through $\mathcal{C} \to [\mathcal{C}]_{\mathrm{ab}}$, i.e. there is a commutative diagram

$$\begin{array}{ccc} [\mathcal{C}]_{\mathrm{ab}} & \overset{\exists!\tilde{\chi}}{\dashrightarrow} & G \\ \uparrow & \nearrow_{\chi} & \\ \mathcal{C} & & \end{array} \tag{3.98}$$

In general, the Grothendieck group will be difficult to describe. But in case of a length category we can give a very explicit description which reflects that the category is built from simple objects.

Theorem 3.5.1. *Let $\mathcal{C}$ be an essentially small length category and let $\{[S_i]\}_{i \in I}$ be the set of isomorphism classes of simple objects. Then the map*

$$\begin{array}{rcl} \mathcal{C} & \to & \mathbb{Z}^I \\ X & \mapsto & \sum_{i \in I} [X : S_i][S_i] \end{array} \tag{3.99}$$

is an additive function and induces an isomorphism

$$[\mathcal{C}]_{\mathrm{ab}} \simeq \mathbb{Z}^I\,, \tag{3.100}$$

i.e. $[\mathcal{C}]_{\mathrm{ab}}$ is the free abelian group with basis the isomorphism classes of simple objects in $\mathcal{C}$.

For the proof we'll need a general lemma about composition series.

Lemma 3.5.2. *Let $\mathcal{C}$ be an abelian category and let $X \in \mathcal{C}$ be an object of finite length. If U is a subobject of X, then X has a composition series in which U appears as a term.*

Proof. Let $0 = X_0 < X_1 < \ldots < X_n = X$ be a composition series of X. Intersecting with U yields a chain

$$0 = U \cap X_0 \leq U \cap X_1 \leq \ldots \leq U \cap X_n = U \tag{3.101}$$

and taking the union with U yields a chain

$$U = U \cup X_0 \leq U \cup X_1 \leq \ldots \leq U \cup X_n = X\,. \tag{3.102}$$

Note that the inequalities are not necessarily strict. Nonetheless, we claim that all quotients in the chains are either simple or 0 so that by removing superfluous terms, the combination of the two series yields a composition series of X in which U appears as a term.

The kernel of the map $X_{i-1} \to X_i \to X_i/(U \cap X_i)$ is $U \cap X_{i-1}$. Hence, $X_{i-1}/(U \cap X_{i-1})$ is a subobject of $X_i/(U \cap X_i)$. We thus get the following commutative diagram of canonical maps with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U \cap X_{i-1} & \longrightarrow & U \cap X_i & \longrightarrow & (U \cap X_i)/(U \cap X_{i-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_{i-1} & \longrightarrow & X_i & \longrightarrow & X_i/X_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_{i-1}/(U \cap X_{i-1}) & \longrightarrow & X_i/(U \cap X_i) & \longrightarrow & (U \cup X_i)/(U \cup X_{i-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.103}$$

If you turn around the diagram by 90 degrees and mirror, you see that we can apply the nine lemma (Exercise 3.2.29) to deduce that there is an exact sequence

$$0 \longrightarrow (U \cap X_i)/(U \cap X_{i-1}) \longrightarrow X_i/X_{i-1} \longrightarrow (U \cup X_i)/(U \cup X_{i-1}) \longrightarrow 0 \,. \tag{3.104}$$

This shows that $(U \cap X_i)/(U \cap X_{i-1})$ is a subobject of X_i/X_{i-1} and that $(U \cup X_i)/(U \cup X_{i-1})$ is a quotient of X_i/X_{i-1}. Since X_i/X_{i-1} is simple, it follows that these two objects are either simple or zero. This is exactly what we claimed. □

Proof of Theorem 3.5.1. For each $i \in I$ let $\chi_i \colon \mathcal{C} \to \mathbb{Z}$ be the map $X \mapsto [X \colon S_i]$. We want to show that χ_i is additive. Let $0 \to X \to Y \to Z$ be an exact sequence. Then $Z \simeq Y/X$. By Lemma 3.5.2 we can find a composition series of Y of the form

$$0 = Y_0 < Y_1 < \ldots < Y_m = X < Y_{m+1} < \ldots < Y_n = Y \,. \tag{3.105}$$

For $j > m$ we have $(Y_j/X)/(Y_{j-1}/X) \simeq Y_j/Y_{j-1}$, which is simple. Hence,

$$0 = Y_m/X < Y_{m+1}/X < \ldots < Y_n/X = Y/X \tag{3.106}$$

is a composition series of Y/X. It is now clear that $\chi_i(Y) = \chi_i(X) + \chi_i(Z)$, i.e. χ_i is additive. Combining the χ_i we get an additive function

$$\begin{array}{ccc} \mathcal{C} & \to & \mathbb{Z}^I \\ X & \mapsto & \sum_{i \in I} [X \colon S_i][S_i] \,. \end{array} \tag{3.107}$$

Let χ denote the induced group morphism $[\mathcal{C}]_{\mathrm{ab}} \to \mathbb{Z}^I$. We need to show that this is an isomorphism. We can give an explicit inverse, namely the map

$$\begin{array}{rcl} \eta \colon \mathbb{Z}^I & \to & [\mathcal{C}]_{\mathrm{ab}} \\ {[S_i]} & \mapsto & [S_i] \,. \end{array} \tag{3.108}$$

First note that since $\mathbb{Z}^I$ is free with basis $\{[S_i]\}_{i \in I}$, we indeed get a well-defined group morphism from this. Now, it's clear that $\chi \circ \eta([S_i]) = [S_i]$, hence $\chi \circ \eta = \mathrm{id}$. To prove that $\eta \circ \chi = \mathrm{id}$, we need to show that for any $X \in \mathcal{C}$ we have

$$[X] = \sum_{i \in I} [X \colon S_i][S_i] \tag{3.109}$$

in $[\mathcal{C}]_{\mathrm{ab}}$. But this is easily obtained inductively from a composition series of X. Namely, if $0 = X_0 < \ldots < X_n$ is a composition series, we have an exact sequence $0 \to X_{n-1} \to X_n \to S_n \to 0$, where $S_n \coloneqq X_n/X_{n-1}$ is simple. Hence, $[X] = [X_{n-1}] + [S_n]$ in $[\mathcal{C}]_{\mathrm{ab}}$. Now, you continue like this. □

CHAPTER 4

Tensor categories

The theory of abelian categories in Chapter 3 was motivated from the point of view of finding categorical formulations of constructions that we know from sets, vector spaces, etc. On the other hand, you can look at the concept of sets and say that this is boring because the elements of a set can't talk to each other. You thus want to lift this concept to a higher level where you replace elements by objects which can talk to each other via morphisms—i.e. you replace sets by categories and a map of sets by a functor between categories. You have thus obtained a **categorification** of the concept of sets! Once you have categorified the concept, you can ask for a **categorification** of a given set X. What does this mean? It means, you want to find a (essentially small) category $\mathcal{C}$ such that when you forget about morphisms, you get back X. More precisely, there should be a—somewhat natural—bijection $[\mathcal{C}] \simeq X$, where $[\mathcal{C}]$ is the set of isomorphism classes of $\mathcal{C}$. Similarly, you can ask about a categorification of a map $f\colon X \to Y$, which should be a functor $F\colon \mathcal{C} \to \mathcal{C}'$ such that the induced map $[F]\colon [\mathcal{C}] \to [\mathcal{C}']$ between isomorphism classes gives back f. The process of taking the isomorphism classes is called **decategorification**. Depending on the context, decategorification may involve some further natural constructions—like taking the Grothendieck group or the Euler characteristic—but all in all this is a well-defined process. But the other way around—categorification—is just an idea and there's nothing well-defined and unique about it!

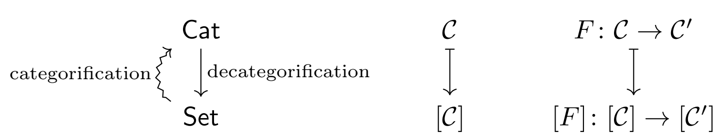


FIGURE 4.1. Categorification and decategorification.

What's the point of this? The point is that sometimes, when you want to prove something about X, things *may* become easier when you know that X actually comes from a category $\mathcal{C}$ because now you can suddenly use arguments with morphisms and *maybe* your question about X is actually implied by something you can prove about objects and morphisms in $\mathcal{C}$. The idea is that you lift things to a higher level and hope that suddenly you can see things more clearly. There is no right and wrong in categorification—you know you're on the right track if you can actually prove something useful with a categorification you constructed. That's why one says:

"Categorification is an art, not a functor."[1]

[1] I actually don't know who first said this. Maybe Ben Webster!?

I claim that all of you have used categorification already without thinking about it. Suppose you notice the identity

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \tag{4.1}$$

for natural numbers k, n. How do you prove this? Maybe you notice that $\frac{n!}{k!(n-k)!}$ is actually the number $\binom{n}{k}$ of k-element subsets of an n-element set—more categorically, it's the number of monomorphisms from a k-element set to an n-element set. From this point of view, the identity above is actually quite easy to prove. Namely, when fixing an element x of your n-element set X then k-element subsets of X can be separated into two cases: those containing x and those not containing x. These are precisely $\binom{n-1}{k-1} + \binom{n-1}{k}$ choices, and this is exactly the identity above. What we have used in this proof was a categorification of the natural numbers $\mathbb{N}$, namely the category set of finite sets! This is admittedly an extremely trivial example but believe me there are (e.g. combinatorial) identities which people were not able to prove directly but managed to do so by using an appropriate categorification. One of the most striking examples is Khovanov's categorification of the Jones knot polynomial in 2000, see [6].

But already the philosophical idea of categorification is really helpful. For example, after realizing that categories are a categorification of sets, you can ask for any concept of (algebraic) structures how to categorify it, and then for a given example you can try to find categorifications. Can you guess what a monoidal category is probably going to be? Can you imagine a categorification of the monoid $(\mathbb{N}, \cdot)$?

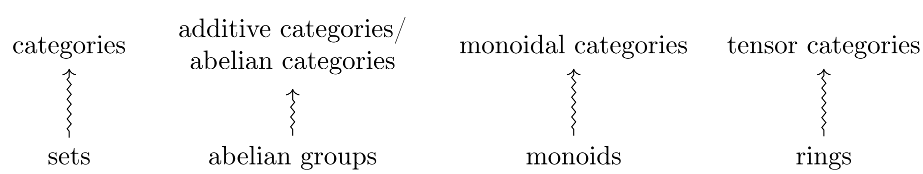


Figure 4.2. Categorifications of basic algebraic structures.

4.1. Monoidal categories

This will eventually be continued. For now, we'll move to [4] for which you have been well-prepared.

Now is not the time for fear. That comes later. — Bane

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