§1. Definitions and first examples

<u>$sl(n)$ is a Lie subalgebra of $gl(n)$</u>: Let $x,y \in sl(n)$, i.e. $Tr(x), Tr(y) = 0$. Then

$$Tr([x,y]) = Tr(xy - yx) = Tr(xy) - Tr(yx) = Tr(xy) - Tr(xy) = 0$$

$$\Rightarrow [x,y] \in sl(n)$$

<u>Adjoint action:</u> L a Lie algebra, $x \in L$. Consider the map

$$\begin{aligned} ad\, x : L &\longrightarrow L \\ y &\longmapsto [xy] \end{aligned}$$

This is a linear map, i.e. $ad\, x \in End_F(L) = gl(L)$.

Claim: $ad\,x$ is a derivation on L, i.e. $(ad\,x)([yz]) = [y\ (ad\,x)(z)] + [(ad\,x)(y)\ z]$

$$\left.\begin{aligned} &LHS: (ad\,x)([yz]) = [x[yz]] \\ &RHS: [y[xz]] + [[xy]z] \end{aligned}\right\} \text{so, claim} \Leftrightarrow [x[yz]] - [y[xz]] - [[xy]z] = 0$$

Use bilinearity and anti-commutativity:

$$-[y[xz]] = [y(-[xz])] = [y[zx]] \quad \text{and} \quad -[[xy]z] = [z[xy]]$$

Hence, claim $\Leftrightarrow$ Jacobi identity $[x[yz]] + [y[zx]] + [z[xy]] = 0$

§2 Ideals and homomorphisms

<u>The center is an ideal</u>: The center of a Lie algebra is $Z(L) = \{z \in L \mid [xz] = 0\ \forall x \in L\}$
Need to show that if $z \in Z(L)$, then $[yz] \in Z(L)\ \forall y \in L$.

$$[x[yz]] \underset{\substack{\uparrow \\ \text{Jacobi}}}{=} -[y\underbrace{[zx]}_{=0}] - [z\underbrace{[xy]}_{=0}] = 0 \Rightarrow [yz] \in Z(L)$$

<u>sl_2 is simple</u>:

$$\begin{aligned} &ad\,x\,(ad\,x\,(ax + by + ch)) \\ &= ad\,x\,([x(ax+by+ch)]) = ad\,x(a[xx] + b[xy] + c[xh]) \\ &= ad\,x\,(0 + bh + c(-2x)) = b[xh] - 2c[xx] = -2bx. \end{aligned}$$

<u>$\exp ad\,x \in Aut\,L$</u>: Let $(A,\cdot)$ be an F-algebra (not necessarily associative). Let $\delta \in End_F(A)$.
Humphreys proved: $(\exp\delta)(x)\cdot(\exp\delta)(y) = (\exp\delta)(x\cdot y)$
$\Rightarrow \exp\delta$ is an algebra automorphism. Now apply to $(A,\cdot) = (L,[\,])$, $\delta = ad\,x$