

§17. Universal enveloping algebra

Recall groups and their representations:

representation $\rho: G \rightarrow GL(V)$ is the same information as a G-module: $G \times V \rightarrow V$ with the obvious properties.

$$\begin{array}{ccc} \text{rep} & & \text{G-mod} \\ \rho: G \rightarrow GL(V) & \rightsquigarrow & G \times V \rightarrow V, (g, v) \mapsto \rho(g)(v) \end{array}$$

$$\begin{array}{ccc} \rho: G \rightarrow GL(V) & \longleftarrow & G \times V \rightarrow V, (g, v) \mapsto gv \\ \rho(g)(v) & := & gv \end{array}$$

A G-module can be considered as an actual module over a ring: the group ring

Let K be our base field/ring. Let KG be the K -algebra with basis $\{\varepsilon_g \mid g \in G\}$ and multiplication:

$$\varepsilon_g \cdot \varepsilon_h := \varepsilon_{gh}.$$

Then any G-module V is naturally a KG -module: $\varepsilon_g v := gv$, and extend linearly. Conversely, any KG -module is naturally a G-module.

\rightarrow can now apply ring and module theory to group representations!

You can do the same with a Lie algebra L . Recall representation $\rho: L \rightarrow \mathfrak{gl}(V) \cong L$ -module V .

Recall this means $\rho([xy]) = [\rho(x), \rho(y)]$, i.e. $[xy]$ acts as the commutator of the action of x and the action of y .

So, in a rep, the bracket is translated into the commutator in a ring.

Can also upgrade this to a module over an honest ring. ^{i.e. associative} But what's the ring?

Tensor algebra of a vector space V is $T(V) = \bigoplus_{i=0}^{\infty} T^i V$; $T^i V = \underbrace{V \otimes \dots \otimes V}_{i\text{-times}}$, with product $(v_1 \otimes \dots \otimes v_n)(w_1 \otimes \dots \otimes w_m) = (v_1 \otimes \dots \otimes v_n \otimes w_1 \otimes \dots \otimes w_m)$

So really, $T(V)$ just consists of formal products of vectors of V with obvious linearity and product relation.

If you take a basis $\{x_i\}_{i \in I}$ of V , then $T(V)$ can be viewed as the non-commutative polynomial ring in the variables x_i ,

$$T(V) \cong K\langle x_i, i \in I \rangle \quad x_i x_j \neq x_j x_i$$

In particular, we have $T(L)$. Now, just do the obvious:

$$U(L) := \frac{T(L)}{\langle xy - yx = [xy] \ \forall x, y \in L \rangle}$$

Then, any L -module V is naturally a $U(L)$ -module and vice versa!

$U(L)$ is the smallest (honest) algebra translating the bracket of L into a commutator in the sense that

we have a map $i: L \rightarrow U(L)$ with $i([xy]) = i(x)i(y) - i(y)i(x)$, and if $i': L \rightarrow U'$ is any other such map then

$$\begin{array}{ccc} L & \xrightarrow{i} & U(L) \\ & \searrow i' & \downarrow \\ & & U' \end{array}$$

That's why we call $U(L)$ the universal enveloping algebra of L .

If L is abelian, then $U(L) = \frac{T(L)}{\langle xy - yx = 0 \ \forall x, y \in L \rangle} = S(L) =$ symmetric algebra of L
 $=$ (commutative) polynomial ring $K[x_i, i \in I]$
 in a basis $\{x_i\}_{i \in I}$ of L

The (commutative) polynomial ring $K[x_1, \dots, x_m]$ has a nice vector space basis: the monomials $x_{i_1} x_{i_2} \dots x_{i_m}$, $i_1 \leq i_2 \leq \dots \leq i_m$, $m \in \mathbb{N}$.

The PBW theorem asserts that the universal enveloping algebra $U(L)$ of in fact any Lie algebra has such a nice basis! The idea is the following. Note that

$$T(V) = \bigoplus_{i=0}^{\infty} T^i(V)$$

is a graded algebra: we have homogeneous components $T^i(V)$ and $T^i(V)T^j(V) \subseteq T^{i+j}(V)$. If $I \subseteq T(V)$ is an ideal generated by homogeneous elements, then $T(V)/I$ inherits the grading and is a graded algebra. Problem, our

$$I = \langle xy - yx - [xy] \mid x, y \in L \rangle \subseteq T(L)$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{deg } 2 & \text{deg } 2 & \text{deg } 1 \end{matrix}$

is not homogeneous. But you can still do the following, for any ideal $I \subseteq T(V)$. First of all, we have a filtration of $T(V)$:

$$F_i(T(V)) := \bigcup_{j \leq i} T^j(V) = \bigoplus_{j \leq i} T^j(V),$$

Any graded algebra is naturally filtered.

so $F_0(T(V)) \subseteq F_1(T(V)) \subseteq \dots$ exhausts all $T(V)$.

Even though $T(V)/I$ may not be graded, it is always filtered:

$$F_i(T(V)/I) = \text{image of } F_i(T(V)) \text{ in } T(V)/I$$

and

$$\text{gr}(T(V)/I) := \bigoplus_i F_i(T(V)/I) / F_{i-1}(T(V)/I) \text{ is a graded algebra: the associated graded .}$$

If I is homogeneous (so $T(V)/I$ is graded), then $T(V)/I \cong \text{gr}(T(V)/I)$ naturally.

Now, let's consider $U(L) = T(L)/I$. Note:

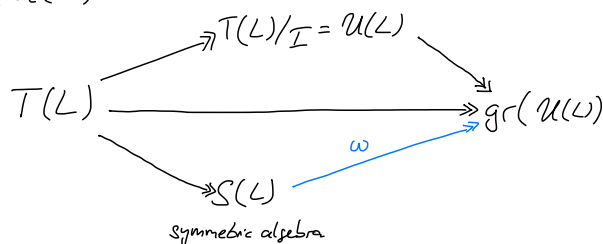
$$\underbrace{xy - yx}_{F_2(T(L)/I)} \equiv \underbrace{[xy]}_{\in F_1(T(L)/I)} \text{ in } T(L)/I$$

The commutator moves downwards in the filtration

so,

$$xy \equiv yx \pmod{F_1(T(L)/I)}$$

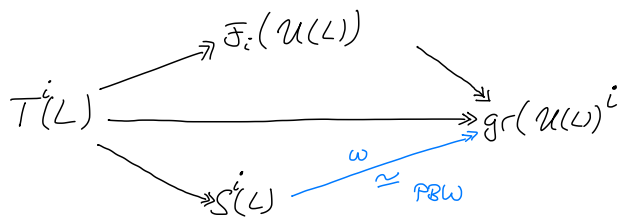
Similarly for higher powers, so $\text{gr}(T(L)/I)$ is commutative, so $T(L) \twoheadrightarrow \text{gr}(U(L))$ descends to $S(L) \xrightarrow{\omega} \text{gr}(U(L))$:



PBW theorem says that ω is also injective, thus an isomorphism.

Why is this good?

Because it implies that $U(L)$ has a vector space basis like $S(L)$!
 Namely, from the above diagram we obtain for each i the diagram



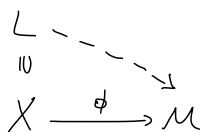
of vector space maps.

If $W \subseteq T^i(L)$ is a subspace that maps isomorphically onto $S^i(L)$ (e.g. space spanned by ordered monomials), then from the upper part of the diagram it follows that the image of W in $\mathbb{F}_i(U(L))$ must be a complement to $\mathbb{F}_{i-1}(U(L))$ (which is killed by $\mathbb{F}_i(U(L)) \rightarrow \mathbb{F}_i(U(L))/\mathbb{F}_{i-1}(U(L)) = S^i(L)$).

So: $U(L)$ has a vector space basis of the form $x_{i_1} \cdots x_{i_m}$, $i_1 \leq \dots \leq i_m, m \in \mathbb{N}$, if $\{x_i\}$ is a basis of L .

In particular, the canonical map $L \rightarrow U(L)$ is injective.

Universal enveloping algebras and PBW theorem allows us to prove existence of free Lie algebras. A Lie algebra L is free on a set $X \subseteq L$ if any map $\phi: X \rightarrow M$ into a Lie algebra M extends uniquely to a Lie algebra morphism $L \rightarrow M$:



For any set X such a thing exists: take the vector space V with basis X and consider $T(V)$ with the commutator as Lie bracket. Let $L \subseteq T(V)$ be the Lie subalgebra generated by X . Let $\phi: X \rightarrow M$ be a map. This extends to a vector space map $V \rightarrow M$. Composition with $M \rightarrow U(M)$ yields $V \rightarrow U(M)$. By property of tensor algebra, this induces algebra morphism $T(V) \xrightarrow{\hat{\phi}} U(M)$. By PBW theorem, we have $M \subseteq U(M)$. By construction, $\hat{\phi}$ maps X to M , hence $\hat{\phi}$ restricts to a (Lie) algebra morphism $L \rightarrow M$, extending ϕ .

Important consequence. Let L be any Lie algebra. Pick a set $X \subseteq L$ of generators. Let $L(X)$ be the free Lie algebra on X . Then $X \hookrightarrow L$ extends to $L(X) \rightarrow L$. This morphism is surjective, hence

$$L(X) / \text{ideal} \simeq L.$$

↑ relations among generators!

Brings us to the question: L semisimple, take as generators sl_2 -triples on a base. What are the relations?