§17. Universal enveloping algebra

Recall groups and their representations:

representation $\rho: G \to GL(V)$ is the same information as a <u>G-module</u>: $G \times V \to V$ with the obvious properties.

$$\begin{array}{ccl} \underline{\text{rep}} & & \underline{\text{G-mod}} \\ \rho: G \to GL(V) & \rightsquigarrow & G \times V \to V, \ (g,v) \mapsto \rho(g)(v) \\ & & \\ \rho: G \to GL(V) & \leftsquigarrow & G \times V \to V, \ (g,v) \mapsto gv \\ \rho(g)(v) := gv & & \end{array}$$

A G-module can be considered as an actual module over a ring: the <u>group ring</u>

Let K be our base field/ring. Let KG be the K-algebra with basis $\{\varepsilon_g \mid g \in G\}$ and multiplication:

$\varepsilon_g \cdot \varepsilon_h := \varepsilon_{gh}$.

Then any G-module V is naturally a KG-module: $\varepsilon_g v := gv$, and extend linearly. Conversely, any KG-module is naturally a G-module.

$\rightsquigarrow$ Can now apply ring and module theory to group representations!

You can do the same with a Lie algebra L. Recall representation $\rho: L \to \mathfrak{gl}(V) \mathrel{\hat{=}} L$-module V.

Recall this means $\rho([xy]) = [\rho(x), \rho(y)]$, i.e. $[xy]$ acts as the commutator of the action of x and the action of y.

So, in a rep, the bracket is translated into the commutator in a ring.

Can also upgrade this to a module over an honest (i.e. associative) ring. But what's the ring?

<u>Tensor algebra</u> of a vector space V is $T(V) = \bigoplus_{i=0}^{\infty} T^i V$; $T^i V = \underbrace{V \otimes \dots \otimes V}_{i\text{-times}}$, with product $(v_1 \otimes \dots \otimes v_k)(w_1 \otimes \dots \otimes w_m) = (v_1 \otimes \dots \otimes v_k \otimes w_1 \otimes \dots \otimes w_m)$

So really, $T(V)$ just consists of formal products of vectors of V with obvious linearity and product relation.

If you take a basis $\{x_i\}_{i \in I}$ of V, then $T(V)$ can be viewed as the <u>non-commutative polynomial</u> ring in the variables x_i,

$$T(V) \simeq K\langle x_i, i \in I\rangle \qquad x_i x_j \neq x_j x_i$$

In particular, we have $T(L)$. Now, just do the obvious:

$$\mathcal{U}(L) := T(L) / \langle xy - yx = [xy] \ \forall x, y \in L \rangle$$

Then, any L-module V is naturally a $\mathcal{U}(L)$-module and vice versa!

$\mathcal{U}(L)$ is the smallest (honest) algebra translating the bracket of L into a commutator in the sense that we have a map $i: L \to \mathcal{U}(L)$ with $i([xy]) = i(x)i(y) - i(y)i(x)$, and if $i': L \to \mathcal{U}'$ is any other such map then

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "i'"'] & \mathcal{U}(L) \arrow[d, dashed] \\ & \mathcal{U}' \end{tikzcd}$$

That's why we call $\mathcal{U}(L)$ the <u>universal enveloping algebra</u> of L.

If L is abelian, then $\mathcal{U}(L) = T(L) / \langle xy - yx = 0 \ \forall x, y \in L \rangle = S(L) =$ symmetric algebra of L

$=$ (commutative) polynomial ring $K[x_i \mid i \in I]$

In a basis $\{x_i\}_{i \in I}$ of L

The (commutative) polynomial ring $K[x_i \mid i \in I]$ has a nice vector space basis: the monomials $x_{i_1} x_{i_2} \cdots x_{i_m}$, $i_1 \leq i_2 \leq \dots \leq i_m$, $m \in \mathbb{N}$.

The PBW theorem asserts that the universal enveloping algebra $\mathcal{U}(L)$ of in fact any Lie algebra has such a nice basis! The idea is the following. Note that

$$T(V) = \bigoplus_{i \geq 0}^{\infty} T^i(V)$$

is a graded algebra: we have homogeneous components $T^i(V)$ and $T^i(V) T^j(V) \subseteq T^{i+j}(V)$.
If $I \subseteq T(V)$ is an ideal generated by homogeneous elements, then $T(V)/I$ inherits the grading and is a graded algebra. Problem, our

$$I = \langle \underset{\uparrow \atop \deg 2}{xy} - \underset{\uparrow \atop \deg 2}{yx} - \underset{\uparrow \atop \deg 1}{[xy]} \mid x, y \in L \rangle \subseteq T(L)$$

is not homogeneous. But you can still do the following, for any ideal $I \subseteq T(V)$. First of all, we have a filtration of $T(V)$:

$$\mathcal{F}_i(T(V)) := \bigcup_{j \leq i} T^i(V) = \bigoplus_{j \leq i} T^i(V),$$

so $\mathcal{F}_0(T(V)) \subseteq \mathcal{F}_1(T(V)) \subseteq \dots$ exhausts all $T(V)$.

Any graded algebra is naturally filtered.

Even though $T(V)/I$ may not be graded, it is always filtered:

$$\mathcal{F}_i(T(V)/L) = \text{image of } \mathcal{F}_i(T(V)) \text{ in } T(V)/I$$

and

$$\mathrm{gr}(T(V)/I) := \bigoplus_i \mathcal{F}_i(T(V)/I) \big/ \mathcal{F}_{i-1}(T(V)/I)$$ is a graded algebra: the associated graded.

If I is homogeneous (so $T(V)/I$ is graded), then $T(V)/I \simeq \mathrm{gr}(T(V)/I)$ naturally.

Now, lets consider $\mathcal{U}(L) = T(L)/I$. Note:

$$\underbrace{xy - yx}_{\mathcal{F}_2(T(L)/I)} \equiv \underbrace{[xy]}_{\in \mathcal{F}_1(T(L)/I)} \text{ in } T(L)/I$$

The commutator moves downwards in the filtration

so,

$$xy \equiv yx \mod \mathcal{F}_1(T(L)/I)$$

Similarly for higher powers, so $\mathrm{gr}(T(L)/I)$ is commutative, so $T(L) \twoheadrightarrow \mathrm{gr}(\mathcal{U}(L))$ descends to $S(L) \overset{\omega}{\twoheadrightarrow} \mathrm{gr}(\mathcal{U}(L))$:

$$\begin{tikzcd}
 & T(L)/I = \mathcal{U}(L) \arrow[dr, two heads] & \\
T(L) \arrow[ur] \arrow[rr, two heads] \arrow[dr] & & \mathrm{gr}(\mathcal{U}(L)) \\
 & S(L) \arrow[ur, two heads, "\omega"] &
\end{tikzcd}$$

symmetric algebra

PBW theorem says that ω is also injective, thus an isomorphism.

Why is this good?

Because it implies that $\mathcal{U}(L)$ has a vector space basis like $S(L)$!
Namely, from the above diagram we obtain for each i the diagram

$$\begin{tikzcd}
 & \mathcal{F}_i(\mathcal{U}(L)) \arrow[dr, two heads] & \\
T^i(L) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dr, two heads] & & gr(\mathcal{U}(L))^i \\
 & S^i(L) \arrow[ur, "\omega", "\simeq\ \mathrm{PBW}"'] &
\end{tikzcd}$$

of vector space maps.
If $W \subseteq T^i(L)$ is a subspace that maps isomorphically onto $S^i(L)$ (e.g. space spanned by ordered monomials), then from the upper part of the diagram it follows that the image of W in $\mathcal{F}_i(\mathcal{U}(L))$ must be a complement to $\mathcal{F}_{i-1}(\mathcal{U}(L))$ (which is killed by $\mathcal{F}_i(\mathcal{U}(L)) \twoheadrightarrow \mathcal{F}_i(\mathcal{U}(L))/\mathcal{F}_{i-1}(\mathcal{U}(L)) = gr(\mathcal{U}(L))^i$.

So: $\mathcal{U}(L)$ has a vector space basis of the form $x_{i_1} \cdots x_{i_m}$, $i_1 \leq \dots \leq i_m$, $m \in \mathbb{N}$, if $\{x_i\}$ is a basis of L.

In particular, the canonical map $L \to \mathcal{U}(L)$ is <u>injective</u>.

Universal enveloping algebras and PBW theorem allows us to prove existence of <u>free</u> Lie algebras.
A Lie algebra L is free on a set $X \subseteq L$ if any map $\phi: X \to M$ into a Lie algebra M extends uniquely to a Lie algebra morphism $L \to M$:

$$\begin{tikzcd}
L \arrow[dr, dashed] & \\
X \arrow[u, phantom, "\cup|"] \arrow[r, "\phi"] & M
\end{tikzcd}$$

For any set X such a thing exists: take the vector space V with basis X and consider $T(V)$ with the commutator as Lie bracket. Let $L \subseteq T(V)$ be the Lie subalgebra generated by X. Let $\phi: X \to M$ be a map. This extends to a vector space map $V \to M$. Composition with $M \to \mathcal{U}(M)$ yields $V \to \mathcal{U}(M)$. By property of tensor algebra, this induces algebra morphism $T(V) \xrightarrow{\tilde{\phi}} \mathcal{U}(M)$. By PBW theorem, we have $M \subseteq \mathcal{U}(M)$. By construction, $\tilde{\phi}$ maps X to M, hence $\tilde{\phi}$ restricts to a (Lie) algebra morphism $L \to M$, extending ϕ.

Important consequence. Let L be any Lie algebra. Pick a set $X \subseteq L$ of generators. Let $L(X)$ be the free Lie algebra on X. Then $X \hookrightarrow L$ extends to $L(X) \to L$. This morphism is surjective, hence

$$L(X)/_{\text{ideal}} \simeq L.$$

relations among generators!

Brings us to the question: L semisimple, take as generators sl_2-triples on a base. What are the relations?