§23 Characters

Let Z be the center of the universal enveloping algebra $\mathcal{U}(L)$ (center as an algebra). Consider the highest weight module $Z(\lambda)$, $\lambda \in H^*$. Let $v^+ \in Z(\lambda)$ be a maximal vector.

<u>Claim:</u> For any $z \in Z$ the element $zv^+ \in Z(\lambda)$ is also a maximal vector.

Proof: If $h \in H^*$, then $h(zv^+) = (hz)v^+ = (zh)v^+ = z(hv^+) = z\lambda(h)v^+ = \lambda(h)(zv^+)$

If $n \in N$, then $n(zv^+) = (nz)v^+ = (zn)v^+ = z(nv^+) = 0$.

□

Since the space of maximal vectors is one-dimensional, zv^+ must be a scalar multiple of v^+, so $zv^+ = \chi_\lambda(z)v^+$ for some $\chi_\lambda(z) \in \mathbb{C}$.
This yields a map $\chi_\lambda : Z \to \mathbb{C}$, the <u>central character</u> of $Z(\lambda)$. It is a $\mathbb{C}$-algebra morphism. The set of all vectors in $Z(\lambda)$ on which z acts by $\chi_\lambda(z)$ is $\mathcal{U}(L)$-stable since z commutes with $\mathcal{U}(L)$. It contains v^+, hence must be all of $Z(\lambda)$ already, i.e. z acts by $\chi_\lambda(z)$ on all of $Z(\lambda)$. It follows that z acts by $\chi_\lambda(z)$ on any submodule and quotient of $Z(\lambda)$. In particular, if $V(\mu)$ is a <u>constituent</u> of $Z(\lambda)$, i.e. is a simple quotient in a composition series of $Z(\lambda)$, then $\chi_\lambda = \chi_\mu$. (Converse needs something stronger).

<u>Question:</u> When is $\chi_\lambda = \chi_\mu$? Is there a combinatorial description?

Yes! Recall the Weyl group $W = \langle \sigma_\alpha \mid \alpha \in \Phi \rangle \subset GL(H^*)$. So, W acts on H^*. We define another action of W on H^*, the <u>dot action</u>, as follows:

$$w \cdot \lambda := w(\lambda + \delta) - \delta \quad , \quad \delta := \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha .$$

Say that $\lambda, \mu \in H^*$ are <u>linked</u> if they lie in the same W-orbit wrt the dot-action.

<u>Claim:</u> If λ, μ are linked, then $\chi_\lambda = \chi_\mu$.

general case not so important

Proof: We assume $\lambda, \mu \in \Lambda$; the general case $\lambda, \mu \in H^*$ follows from Zariski density of Λ in H^*. It is enough to show this in the case $\mu = \sigma_\alpha \cdot \lambda = \sigma_\alpha(\lambda+\delta) - \delta$ for $\alpha \in \Delta$.
We have $\sigma_\alpha \delta - \delta = -\alpha$. Hence,

$$\mu = \sigma_\alpha(\lambda+\delta) - \delta = \sigma_\alpha\lambda + \sigma_\alpha\delta - \delta = \sigma_\alpha\lambda - \alpha = \lambda - (\langle\lambda,\alpha\rangle + 1)\alpha .$$

Since $\lambda \in \Lambda$, we have $m := \langle \lambda, \alpha \rangle \in \mathbb{Z}$. Thinking a bit how $\mathcal{U}(L)$ acts on powers of y_α you can see:

- if m is non-negative, then the image of y_α^{m+1} in $Z(\lambda)$ is a maximal vector of weight $\lambda - (m+1)\alpha = \mu$. Hence, $Z(\mu) \subset Z(\lambda)$, so $\chi_\lambda = \chi_\mu$.

- if m is negative, then $\langle \mu, \alpha \rangle = \langle \lambda, \alpha \rangle - 2(\langle \lambda, \alpha \rangle + 1) = -\langle \lambda, \alpha \rangle - 2$ is non-negative (except for if $\langle \lambda, \alpha \rangle = -1$, but then $\mu = \lambda$ anyways) and by same argument $Z(\lambda) \subset Z(\mu)$, so $\chi_\lambda = \chi_\mu$.

□

In fact, the converse also holds. This requires more work and a better understanding of the center (Harish-Chandra).
Define a linear map $\xi : \mathcal{U}(L) \to \mathcal{U}(H)$ as follows. By PBW, $\mathcal{U}(L)$ has a vector space based formed by the elements

$$\prod_{\alpha \in \Phi^+} y_\alpha^{i_\alpha} \cdot \prod_{\alpha \in \Delta} h_\alpha^{k_\alpha} \cdot \prod_{\alpha \in \Phi^+} x_\alpha^{j_\alpha}$$

Now map basis monomials in the h_α to itself and kill all other basis vectors. This is our linear map $\xi: \mathcal{U}(L) \to \mathcal{U}(H)$. Now, think about how a basis monomial as above acts on a maximal vector $v^+ \in Z(\lambda)$. It's killed by the y_α; the h_α act via χ_λ, and the x_α map to lower weight vectors. From this you deduce that

$$\chi_\lambda(z) = \lambda(\xi(z)), \quad z \in Z$$

From this in turn you deduce that $\xi: Z \to \mathcal{U}(H)$ is an algebra morphism (<u>Harish-Chandra morphism</u>)
Note that $\mathcal{U}(H) \simeq \mathrm{Sym}(H) \simeq \mathbb{C}[h_\alpha \mid \alpha \in \Delta]$ as algebras since H is abelian.
Let us take into account the ρ-shift involved in the dot action by composing ξ with the automorphism $\eta: \mathrm{Sym}(H) \to \mathrm{Sym}(H)$ mapping p to $p(\lambda - \delta)$. Denote the resulting algebra morphism $Z \to \mathrm{Sym}(H)$ by ψ and call it <u>twisted Harish-Chandra morphism</u>.

Then

$$\chi_\lambda(z) = \lambda \circ \xi(z) = (\lambda \circ \xi)(z) = (\lambda \circ \eta^{-1} \circ \psi)(z) = (\lambda \circ \eta^{-1})(\psi(z)) = (\lambda + \delta)(\psi(z))$$

Suppose that λ and μ are linked, i.e. $\mu = w \cdot \lambda$. Then

$$\chi_\lambda(z) = \chi_\mu(z) \Rightarrow (\lambda + \delta)(\psi(z)) = (\mu + \delta)(\psi(z)) = (w \cdot \lambda + \delta)(\psi(z))$$
$$= (w(\lambda + \delta) - \delta + \delta)(\psi(z)) = w(\lambda + \delta)(\psi(z))$$

$\Rightarrow w(\lambda + \delta)$ takes the same value at $\psi(z)$ $\forall w \in W$
$\Rightarrow \lambda + \delta$ takes the same value at $w\psi(z)$ $\forall w \in W$

This is true for all $\lambda \in H^*$ $\Rightarrow$ $w\psi(z)$ must be the same for all $w \in W$
$\Rightarrow \psi(z)$ is fixed by W
$\Rightarrow \psi(z) \in \mathrm{Sym}(H)^W$.

So, the twisted HC morphism is a morphism

$$\psi: Z \to \mathrm{Sym}(H)^W.$$

What we wanted to show is that $\chi_\lambda = \chi_\mu \Rightarrow \lambda, \mu$ linked. This is of course equivalent to λ, μ not linked $\Rightarrow \chi_\lambda \neq \chi_\mu$.

An elementary argument shows: if λ, μ not linked $\to$ λ, μ take distinct values at some element of $\mathrm{Sym}(H)^W$.

So, if we could show that $\psi(Z) = \mathrm{Sym}(H)^W$, then there is $z \in Z$ such that $\lambda(\psi(z)) \neq \mu(\psi(z))$
$\Rightarrow (\lambda + \delta)(\psi(z)) \neq (\mu + \delta)(\psi(z)) \Rightarrow \chi_\lambda(z) \neq \chi_\mu(z) \Rightarrow \chi_\lambda \neq \chi_\mu$.

Theorem of HC is now basically about showing that $\psi(Z) = \mathrm{Sym}(H)^W$.

One can furthermore show that ψ is injective, so $Z \simeq \mathrm{Sym}(H)^W$.