§3/4 Solvable and nilpotent Lie algebras, Theorems of Lie and Cartan

How can we get some organization into the whole class of Lie algebras, or into a given Lie algebra?

Solvable Lie algebras (§3.1)

If L is abelian, i.e. $[L,L]=0$, there is not much to understand (boring/easy).
Let's build a more complicated class out of abelian Lie algebras.
An important concept is that of an extension of Lie algebras. Suppose we have a short exact sequence

$$0 \to L' \xrightarrow{f} L \xrightarrow{g} L'' \to 0$$

of Lie algebras, i.e. Im = Ker, so $L'' \cong L/L'$. Then one says L is an extension of L'' by L'.
So, in a sense, L is obtained by "gluing" L' and L''. An easy way to glue is taking the direct sum:

$$0 \to L' \to L' \oplus L'' \to L'' \to 0$$

In general, gluing can be quite complicated!

Now, define the class S of solvable Lie algebras as the smallest class of Lie algebras containing the abelian Lie algebras and which is closed under forming extensions, i.e. if $L', L'' \in S$ and L is an extension of L'' by L', then $L \in S$.
This class exists and can be constructed as follows:
- $S_0 :=$ abelian Lie algebras
- $S_i :=$ extensions of a Lie algebra $\in S_{i-1}$ by an abelian Lie algebra.

Then $S = \bigcup_{i \in \mathbb{N}} S_i$. So, you see a hierarchy in solvability.

If we "expand out" this inductive definition and use isomorphism theorems, we see that for a solvable Lie algebra we have a sequence of extensions

$$0 \to \underset{\text{abelian}}{L_1} \to L \to \underset{\text{solvable}}{L/L_1} \to 0$$

$$0 \to \underset{\text{abelian}}{L_2/L_1} \to L/L_1 \to \underset{\text{solvable}}{L/L_2} \to 0$$

$$0 \to \underset{\text{abelian}}{L_3/L_2} \to L/L_2 \to \underset{\text{solvable}}{L/L_3} \to 0$$

etc.

so, this corresponds to a filtration

$$L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \dots \supseteq L_n = 0$$

such that L_i/L_{i-1} is abelian $\forall i$.

Now, note: If $L' \subseteq L$ is an ideal such that L/L' is abelian, then $[L,L] \subseteq L'$.
Hence, in the filtration, $L_i \supseteq [L_{i-1}, L_{i-1}]$. We can thus refine the filtration by throwing the $[L_i, L_i]$ in between. Hence, the derived series goes all the way down to 0
$\Rightarrow L$ solvable as defined by Hum ($\Leftarrow$ also clear).

Okay, now we have a special class of Lie algebras. Given any Lie algebra L, Hum has shown that there is a unique maximal solvable ideal, denoted Rad L.

$$0 \to \underset{\text{solvable}}{\mathrm{Rad}\,L} \to L \to \underset{\text{semisimple}}{L/\mathrm{Rad}\,L} \to 0$$

Hence, any Lie algebra is an extension of a semisimple ($\Leftrightarrow$ no non-zero solvable ideal) Lie algebra by a solvable Lie algebra.

We will be able to classify semisimple Lie algebras $/\mathbb{C}$; classifying solvable algebras is hopeless! Anyways, there are some bits we can understand about solvable algebras, and this is also useful for semisimple algebras.

<u>Primary example of a solvable algebra?</u> The algebra $\mathfrak{t}$ of upper triangular matrices.
Basis-free version: linear operators $f \in \mathfrak{gl}(V)$ preserving a fixed flag $0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$, i.e. $f(V_i) \subseteq V_i$ $\forall i$. This is a general property of solvable Lie algebras.

<u>Lie's theorem (§4.1)</u>
If $L \subseteq \mathfrak{gl}(V)$ is solvable, then L stabilizes a flag in V, i.e. there is

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V, \quad \dim V_i = i, \quad L.V_i \subseteq V_i \ \forall i$$

$\Rightarrow$ V has a basis such that L consists of upper triangular matrices.

Note: if L is an (abstract) solvable Lie algebra and $\rho: L \to \mathfrak{gl}(V)$ is a representation, then, since $\rho(L) \subseteq \mathfrak{gl}(V)$ is solvable, L stabilizes a flag in V. $\rightsquigarrow$ <u>a theorem about representations of solvable Lie algebras.</u>

<u>Nilpotent Lie algebras (§3.2)</u>
Here is a refinement of solvability. An extension $0 \to L' \to L \to L'' \to 0$ is called <u>central</u> if L' is contained in the center of L. In particular, L' is abelian. Similarly as above, define the class $\mathcal{N}$ of nilpotent Lie algebras as the smallest class containing all abelian algebras and which is closed under <u>central</u> extensions. Clearly, $\mathcal{N} \subseteq \mathcal{S}$.

As above, for a nilpotent Lie algebra we have a sequence of central extensions

$$0 \to \underset{\text{abelian}}{L_1} \xrightarrow{\text{central}} L \to \underset{\text{nilpotent}}{L/L_1} \to 0$$

$$0 \to \underset{\text{abelian}}{L_2/L_1} \xrightarrow{\text{central}} L/L_1 \to \underset{\text{nilpotent}}{L/L_2} \to 0$$

$$0 \to \underset{\text{abelian}}{L_3/L_2} \xrightarrow{\text{central}} L/L_2 \to \underset{\text{nilpotent}}{L/L_3} \to 0$$

etc.

so, this corresponds to a filtration

$$L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \ldots \supseteq L_n = 0$$

such that L_i/L_{i-1} is central in L/L_{i-1}.

L_i/L_{i-1} being central in L/L_{i-1} means that $[L_i/L_{i-1}, L/L_{i-1}] = 0$, in other words, $[L_i, L] \subseteq L_{i-1}$. We can thus always throw the terms $[L_i, L]$ in the filtration and see that Hum's definition of nilpotency is exactly the same.

<u>Primary example of a nilpotent algebra?</u> The algebra $\mathfrak{n}$ of strictly upper triangular matrices. Basis-free version: linear operators $f \in \mathfrak{gl}(V)$ not only stabilizing a fixed flag $0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = V$ but shifting down: $f(V_i) \subseteq V_{i-1}$.

This is also a general property of nilpotent algebras. <u>Well, almost</u>: the algebra $\mathfrak{d}$ of diagonal matrices is nilpotent (since it is abelian) but of course it does not shift down in a flag.

So, we have to assume more.

Engel's theorem (§32)

If $L \subseteq gl(V)$ is a subalgebra consisting of nilpotent endomorphisms then there is a flag $0 = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V$ with $L.V_i \subseteq V_{i-1}$ $\forall i$. $\Rightarrow$ V has a basis such that L consists of strictly upper triangular matrices.

Hence: Suppose L is an arbitrary Lie algebra such that all $x \in L$ are ad-nilpotent. Then $\mathrm{ad}L \subseteq gl(L)$ consists of nilpotent endomorphisms. Hence, there is $0 = L_0 \subset L_1 \subset \dots$ with $L.L_i \subseteq L_{i-1}$. Let $0 \neq x \in L_1$. Then $L.L_1 \subseteq L_0 = 0$ means $[L,x] = 0 \Rightarrow x \in Z(L) \Rightarrow Z(L) \neq 0$. Have extension $0 \to Z(L) \to L \to L/Z(L) \to 0$. $L/Z(L)$ has smaller dim than L and consists of ad-nilp elements. $\rightsquigarrow$ inductively, L nilpotent. So nilpotent $\Leftrightarrow$ ad-nilpotent.

Remark: if $L \subseteq gl(V)$ consists of nilpotent endomorphisms, it is ad-nilpotent, hence nilpotent.