§5 Killing form

Corollary 5.2

L semisimple $\Rightarrow L = L_1 \oplus \dots \oplus L_t$ with simple ideals L_t. Hence, any non-zero quotient is semisimple, hence non-abelian $\Rightarrow$ Since $L/[LL]$ is abelian $\Rightarrow L/[LL] = 0 \Rightarrow L = [LL]$.

§6. Complete reducibility of representations

Schur's Lemma: If V is an irreducible L-module, then every non-zero endomorphism $f: V \to V$ of L-modules is already an isomorphism. Moreover, if F is algebraically closed, then $\mathrm{End}_L(V) = F$.

Proof: Let $f: V \to V$ be a non-zero morphism. Since $\mathrm{Ker} f \subsetneq V$ is a submodule and V is irreducible, must have $\mathrm{Ker} f = 0$. Similarly, $\mathrm{Im} f = V \Rightarrow f$ is an isomorphism.
If F is algebraically closed, f has an eigenvalue $\lambda \in F$. The map $f - \lambda \cdot id$ is a morphism $V \to V$. It is not an isomorphism (has non-trivial kernel), hence $f - \lambda \cdot id = 0$ by first part $\Rightarrow f = \lambda \cdot id$. $\square$

Weyl's Theorem

Key claim: Let V be a faithful L-module. Suppose there is an irreducible submodule W of V of codimension 1. Then V is completely reducible.
Proof: The Casimir Operator $c := c_V$ of V is an L-module endomorphism on V. Hence, $\mathrm{Ker}\, c$ is a submodule of V. We will show that $V = W \oplus \mathrm{Ker}\, c$.
It is enough to show that $W \cap \mathrm{Ker}\, c = 0$ and $\mathrm{Ker}\, c \neq 0$ (since W of codimension 1).

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] & V \arrow[r] \arrow[loop above, "c := c_V"] & V/W \arrow[r] & 0 \end{tikzcd}$$

We know that $\mathrm{Tr}\, c = \dim L \neq 0$.
V/W is a 1-dimensional L-module. Hence, L acts trivially on V/W. Hence, also c acts trivially on V/W by construction of c, so $\mathrm{Tr}(c \text{ on } V/W) = 0$. Since $\mathrm{Tr}\, c \neq 0$, must have $\mathrm{Tr}(c|_W) \neq 0$. Since W is irreducible, c acts by a scalar on W. This scalar must be non-zero since $\mathrm{Tr}(c|_W) \neq 0$. Hence $W \cap \mathrm{Ker}\, c = 0$. Moreover, $\mathrm{Ker}\, c$ cannot be $= 0$ since then c could not act trivially on V/W.

$\square$