§15+16: Cartan subalgebras and conjugacy theorems

We're almost done with the classification of finite-dimensional semisimple Lie algebras.
We still need to prove existence (only for the exceptional types G_2, F_4, E_6, E_7, E_8).
But befor that we need to address one delicate issue:
We only get a root system for a ss. Lie algebra L after choosing a maximal toral subalgebra H.
In principle it could happen that another choice H' yields another (non-isomorphic root) system.
Note: when I (quite sloppy) said that if $\phi: L \to L'$ is an isomorphism of Lie algebras, then L and L' have isomorphic root systems, this was with respect to the choices $H \subset L$ and $\phi(H) \subset L'$!
So, my statement was more precisely that an isomorphism $\phi: (L, H) \to (L', H')$ of pairs ($\phi(H) = H'$) induces an isomorphism $\Phi(L,H) \to \Phi(L', H')$.

We always have $\dim L = \operatorname{rank} \Phi + \operatorname{card} \Phi$, so the root systems for different choices cannot be totally random
But this is not enough: e.g. B_ℓ and C_ℓ have same dim and card, but are non-isomorphic.

One feature that would solve this would be if any two maximal toral subalgebras $H, H' \subset L$ are conjugate via some $\sigma \in \operatorname{Aut}(L)$, i.e. $\sigma(H) = H'$. Then $\Phi(L,H) \simeq \Phi(L,H')$ really just depends on L.
This is in fact true. I think, this is actually the most difficult part in the classification!

The basic idea is as follows. Recall that a maximal toral subalgebra H is abelian. Hence, it is solvable, and is thus contained in a maximal solvable subalgebra $B \subset L$. Such an algebra is called a Borel subalgebra.
One can actually give some very specific such B: choose a base Δ of $\Phi(L,H)$ and set

$$B(\Delta) := H \oplus \underbrace{\bigoplus_{\alpha \in \Phi^+} L_\alpha}_{=: N(\Delta)}$$

This is a subalgebra and $[B(\Delta), B(\Delta)] = N(\Delta)$ by root space properties. Moreover, $N(\Delta)$ is nilpotent: if $x \in L_\alpha$, $\alpha \in \Phi^+$, then application of $\operatorname{ad} x$ to root vectors for roots of positive height increases height by at least one; this makes the descending central series go to zero. Hence $B(\Delta)$ is solvable. Let K be any subalgebra of L properly containing $B(\Delta)$. Then K, being stable under $H \subset K$, must include some L_α for $\alpha \in \Phi^-$. But then K contains the $\mathfrak{sl}_2$-algebra S_α, so K is not solvable ↯ $\Rightarrow B(\Delta)$ is a Borel.

Now, inside B, the subalgebra H is toral, but not necessarily maximal toral. The key ideas are the following:
1. Introduce notion of Cartan subalgebra in an arbitrary Lie algebra L: a nilpotent self-normalizing subalgebra.
2. Existence of such a thing is a priori unclear but: show that a subalgebra is a Cartan subalgebra iff it is an algebra minimal among those of the form $L_0(\operatorname{ad} x)$, $x \in L$ (minimal Engel subalgebra; call such an x regular element)
   generalized eigenspace of $\operatorname{ad} x$ for eigenvalue 0, i.e. $\operatorname{Ker}(t - \operatorname{ad} x)^m$ for m large enough
   $\Rightarrow$ every Lie algebra has a Cartan subalgebra.
3. In a semisimple Lie algebra: maximal toral = Cartan!
4. Inside a solvable algebra L all Cartan subalgebras are conjugate.
   Even with respect to special automorphisms, namely those in the subgroup $\mathcal{E}(L) \subset \operatorname{Int} L$ generated by $\exp \operatorname{ad} x$ for $x \in L$ strongly ad-nilpotent, i.e. there is $y \in L$ such that $x \in L_a(\operatorname{ad} y)$ for some nonzero a ($\Rightarrow x$ ad-nilpotent)
   Why $\mathcal{E}(L)$? Because it behaves nicely wrt subalgebras: if $K \subset L$ and $x \in K$ strongly ad-nilpotent, then $x \in L$ strongly ad-nilpotent since $K_a(\operatorname{ad} y) \subseteq L_a(\operatorname{ad} y)$. Let $\mathcal{E}(L;K) \subseteq \mathcal{E}(L)$ be the corresponding

subgroup. Then $\mathcal{E}(K)$ is obtained simply by restricting the $\sigma \in \mathcal{E}(L;K)$ to K.

5. Show that all Borel subalgebras of a Lie algebra L are conjugate under $\mathcal{E}(L)$

   This makes use of 4 already!

6. All Cartan subalgebras of a Lie algebra are conjugate under $\mathcal{E}(L)$: let $H, H' \subset L$ be CSAs of L.
   Since they are nilpotent, they are solvable, hence $H \subset B$, $H' \subset B'$ for Borel subalgebras B, B'.
   Note: $N_B(H) = N_L(H) \cap B = H \cap B = H$, so H is a CSA of B. Similarly $H' \subset B'$.
   By 5) $\exists \sigma \in \mathcal{E}(L)$ s.t. $\sigma(B) = B'$. Now, $\sigma(H)$ and H' are both CSA's of the solvable algebra B'.
   Hence by 4 $\exists \tau' \in \mathcal{E}(B')$ s.t. $\tau'\sigma(H) = H'$! But τ' is the restriction to B' of some $\tau \in \mathcal{E}(L;B') \subset \mathcal{E}(L)$, so
   $\tau\sigma(H) = H'$ for $\tau\sigma \in \mathcal{E}(L)$.

$\square$ Party