Examples and counterexamples in sheaf theory

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[1] https://ulthiel.com/math

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Constant presheaf is not a sheaf

If A is an abelian group, the constant presheaf on a space X is the presheaf $\mathcal{F}$ with $\mathcal{F}(U) = A$ for all U. This is not a sheaf since the value of a sheaf on the empty set needs to be the zero group. Generally, the problem is that if U_1 and U_2 are two disjoint open subsets, then taking constant functions C_i on U_1 and U_2 with different value there is no constant function C on $U_1 \cup U_2$ with both $C|_{U_1} = C_1$ and $C|_{U_2} = C_2$. The sheafification of $\mathcal{F}$ is the sheaf $\underline{A}$ of *locally* constant functions on X, which eliminates this problem.

Presheaf tensor product is not a sheaf

This is from https://math.stackexchange.com/q/1488296/262462

If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves, the presheaf tensor product $\mathcal{P}(U) = \mathcal{F}(U) \otimes_{\mathbb{Z}} \mathcal{G}(U)$ is in general not a sheaf. Here is a counterexample. Let X be a topological space and let $\mathcal{F} = \mathcal{G} = \underline{\mathbb{Z}}$ be the sheaf of locally constant functions $X \to \mathbb{Z}$. Since locally constant functions are determined by their (constant) values on the connected components, we have $\mathcal{F}(U) = \mathbb{Z}^{n_U}$, where n_U is the number of connected components of U. Hence,

$$\mathcal{P}(U) = \mathcal{F}(U) \otimes_{\mathbb{Z}} \mathcal{G}(U) = \mathbb{Z}^{n_U} \otimes_{\mathbb{Z}} \mathbb{Z}^{n_U} \simeq \mathbb{Z}^{n_U^2} \;,$$

the latter isomorphism following from the fact that the tensor product of two free abelian groups is free of rank equal to the product of the ranks of the factors. In particular, if U is connected, then $\mathcal{P}(U) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$. Suppose that $\mathcal{P}$ is a sheaf. If $U_1, \ldots, U_{n_U}$ are the connected components of U, then since connected components are disjoint, the sheaf axioms force

$$\mathcal{P}(U) = \mathcal{P}(U_1) \oplus \ldots \oplus \mathcal{P}(U_{n_U}) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} = \mathbb{Z}^{n_U} \;.$$

If $n_U > 1$, then $\mathbb{Z}^{n_U}$ and $\mathbb{Z}^{n_U^2}$ are not isomorphic since they are two free abelian groups of different ranks n_U and n_U^2.

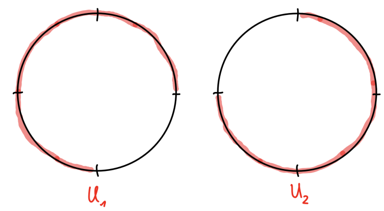
Presheaf quotient is not a sheaf

This is from https://math.stackexchange.com/q/1466781

If $\mathcal{G}$ is a sheaf and $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, the presheaf quotient $\mathcal{P}(U) = \mathcal{G}(U)/\mathcal{F}(U)$ is in general not a sheaf. Here is a counterexample. Let S^1 be the unit circle and consider

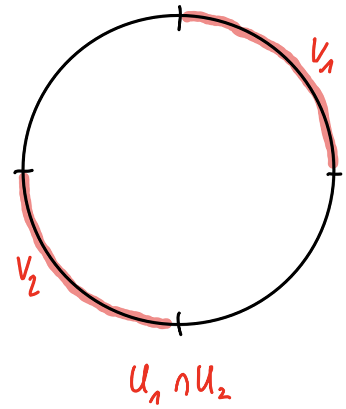
$$p \colon \mathbb{R} \to S^1 \;, \quad t \mapsto (\cos(2\pi t),\, \sin(2\pi t)) \;.$$

This map is a covering map. Let $\mathcal{G}$ be the sheaf of continuous functions $S^1 \to \mathbb{R}$ and let $\mathcal{F}$ be the subsheaf of locally constant functions $S^1 \to \mathbb{R}$. We will now cover S^1 by two overlapping open subsets where one part will involve another sheet of the covering which will cause a problem: let U_1 be the image of the interval $I_1 := (0, \frac{3}{4})$, giving the first three quarters of the circle, and let U_2 be the image of the interval $I_2 := (\frac{1}{2}, 1\frac{1}{4})$, giving the second half plus the first quarter.



The overlap $U_1 \cap U_2$ is the disjoint union $V_1 \coprod V_2$ with V_1 being the first quarter and V_2 being the third quarter.

The restriction $p|_{I_i} \colon I_i \to U_i$ has a continuous inverse $f_i \colon U_i \to I_i$. So, $f_i \in \mathcal{G}(U_i)$. We have $f_2|_{V_2} = f_1|_{V_2}$ but $f_2|_{V_1} = f_1|_{V_1} + 1$. Hence, f_1 and f_2 do not agree on the overlap $U_1 \cap U_2$. But the difference is the locally constant map on $U_1 \cap U_2$ which is 0 on V_2 and 1 on V_1, so



$$f_1|_{U_1 \cap U_2} \equiv f_2|_{U_1 \cap U_2} \bmod \mathcal{F}(U_1 \cap U_2) \ .$$

Suppose that $\mathcal{P}$ is a sheaf. Then the above equation implies that there is a (unique) function $f \in \mathcal{G}(S^1)$ such that $f|_{U_i} \equiv f_i \bmod \mathcal{F}(U_i)$. This means there is $C_i \in \mathcal{F}(U_i)$ such that

$$f|_{U_i} = f_i + C_i \ .$$

Note that since U_i is connected, the function C_i is constant on all of U_i. We now restrict the previous equation separately to V_1 and V_2. On the one hand, we have

$$f_1|_{V_1} + C_1 = f|_{V_1} = f_2|_{V_1} + C_2 \Longrightarrow f_1|_{V_1} + V_1 = f_1|_{V_1} + 1 + C_2 \Longrightarrow C_1 = 1 + C_2 \ .$$

On the other hand, we have

$$f_1|_{V_2} + C_1 = f|_{V_2} = f_2|_{V_2} + C_2 \Longrightarrow f_1|_{V_2} + C_1 = f_1|_{V_2} + C_2 \Longrightarrow C_1 = C_2 \ .$$

This is a contradiction.

Surjective sheaf morphism is not surjective on open sets

If $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves on a space X, meaning that $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$, then $\varphi(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is not surjective in general. Here is a counterexample.

This is from `https://math.stackexchange.com/q/58306`

Let $X = \mathbb{C}$ equipped with the Zariski topology. Then all open subsets of U have non-empty intersection, in particular any open subset is connected. The constant sheaf $\mathcal{F} = \underline{\mathbb{Z}}$ on X thus has sections $\mathcal{F}(U) = \mathbb{Z}$ for any open subset U. For a point $x \in X$ let $\mathcal{G}^x$ be the skyscraper sheaf at x with value $\mathbb{Z}$. Recall that this is defined by

$$\mathcal{G}^x(U) = \begin{cases} \mathbb{Z} & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

with the obvious restriction maps. The stalk $\mathcal{G}^x_y$ of $\mathcal{G}^x$ in a point y is

$$\mathcal{G}^x_y = \begin{cases} \mathbb{Z} & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

We have a sheaf morphism $\varphi^x \colon \mathcal{F} \to \mathcal{G}^x$ where $\varphi(U) \colon \mathcal{F}(U) \to \mathcal{G}^x(U)$ is the identity if $x \in U$ and is the zero map otherwise.

Now, pick two distinct points $P \neq Q$ in X. Let $\mathcal{G} = \mathcal{G}^P \oplus \mathcal{G}^Q$. We then have the sheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ with $\varphi(U) = \varphi^P(U) \oplus \varphi^Q(U)$. Since P and Q are distinct points, the stalks of $\mathcal{G}$ are

$$\mathcal{G}_x = \begin{cases} \mathbb{Z} & \text{if } x = P, Q \\ 0 & \text{otherwise.} \end{cases}$$

This shows that φ_x is surjective for any $x \in X$, so φ is a surjective sheaf moprhism. But φ is not surjective on open sets since on $U = X$ we have $\varphi(X) \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $z \mapsto (z, z)$, which is not surjective.