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REAL REPRESENTATIONS OF CYCLIC GROUPS

Let G be a cyclic group of order n.

Fix a generator $g \in G$ and let $\zeta := e^{\frac{2\pi i}{n}} \in \mathbb{C}$.

For $0 \leq k < n$ let

$$\rho_k : G \to \mathbb{C}^\times = GL_1(\mathbb{C}), \quad g \mapsto \zeta^k.$$

These are representation of G.

Clearly, they are irreducible and pairwise non-isomorphic.

They give all the irreducible complex representations since

$$\#\{\rho_k\} = n = \#G = \#\text{conjugacy classes}$$

But what about the real representations of G?

First, note that $\mathbb{R}G$ is still semisimple by Maschke's theorem, so every every real representation of G is a direct sum of irreducible real representations.

The only ρ_k which are defined over $\mathbb{R}$ are:

- $\rho_0 = \mathrm{triv}$
- $\rho_{n/2}$ if n is even (g acting by -1)

Is there anything else?

Let $\mathbb{C}_\mathbb{R}$ be $\mathbb{C}$ considered as a real vector space with basis $\{1, i\}$.

Let $\varphi_k := \frac{2\pi k}{n}$ for $0 \leq k < n$. We have

$$\varsigma^k \cdot 1 = e^{i\varphi_k} = \cos(\varphi_k) + i \cdot \sin(\varphi_k)$$
$$\Rightarrow \varsigma^k \cdot i \qquad = -\sin(\varphi_k) + i\cos(\varphi_k)$$

Hence, multiplication $\mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ by ς^k is given by the matrix

$$\begin{pmatrix} \cos(\varphi_k) & -\sin(\varphi_k) \\ \sin(\varphi_k) & \cos(\varphi_k) \end{pmatrix} =: R_k$$ Rotation through angle φ_k

Since $(\varsigma^k)^n = 1 \Rightarrow R_k^n = 1$, hence

$$\psi_k : G \longrightarrow GL(\mathbb{C}_{\mathbb{R}}) \simeq GL_2(\mathbb{R}),\ g \mapsto R_k,$$

is a 2-dimensional real representation of G.

Note: $\varsigma^k \in \mathbb{R} \overset{0 \leq k < n}{\Leftrightarrow} k = 0$ or $k = \frac{n}{2}$ if n is even. For these cases we have:

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \psi_0 \simeq \varsigma_0 \oplus \varsigma_0 \quad \text{(2 copies of triv)}$$

and

$$\text{if } n \text{ even} \Rightarrow \varphi_{n/2} = \pi \Rightarrow R_{n/2} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \psi_{n/2} = \varsigma_{n/2} \oplus \varsigma_{n/2}$$

<u>Lemma:</u> If $\varsigma^k \notin \mathbb{R}$, then ψ_k is irreducible over $\mathbb{R}$.

Proof: Assume ψ_k is not irreducible. Then ψ_k would have ς_0 or $\varsigma_{n/2}$ as subrepresentation (since $\dim \psi_k = 2$ and $\varsigma_0, \varsigma_{n/2}$ are the only 1-dimensional real representations). This means, there is a subspace of $\mathbb{C}_{\mathbb{R}}$ where g acts by ± 1. But g acts by $\varsigma^k \notin \mathbb{R}$ on all of $\mathbb{C}_{\mathbb{R}}$. ↯. ■

Let χ_{ψ_k} be the character of ψ_k. Then

$$\chi_{\psi_k}(g) = \operatorname{tr} R_k = 2\cos(\varphi_k) = \zeta^k + \overbrace{\overline{\zeta^k}}^{=\zeta^{-k}} = \chi_{\rho_k} + \overline{\chi_{\rho_k}}$$

Hence, the extension $\psi_k^{\mathbb{C}}$ of ψ_k to $\mathbb{C}$ decomposes as

$$\psi_k^{\mathbb{C}} \simeq \rho_k \oplus \overline{\rho_k}$$

Indeed,

$$A R_k A^{-1} = \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix} \quad \text{with} \quad A := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in GL_2(\mathbb{C})$$

Now, there is the following general fact, see Lam (1991):

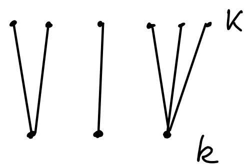
finite-dimensional

(7.13) Proposition. *Let R be a k-algebra and $K \supseteq k$ be a field extension. Then:*

(1) *any simple left R^K-module V is a composition factor of M^K for some simple left R-module M; and*

(2) *if M_1, M_2 are non-isomorphic simple left R-modules, then M_1^K and M_2^K cannot have a common composition factor.*

Visualization



For n odd we have the following picture when extending from $\mathbb{R}$ to $\mathbb{C}$:

$\mathbb{C}$	ρ_0	$\rho_0 \oplus \rho_0$	$\rho_1 \oplus \rho_{n-1}$	$\dots$	$\rho_{\frac{n-1}{2}} \oplus \rho_{\frac{n+1}{2}}$	$\rho_{\frac{n+1}{2}} \oplus \rho_{\frac{n-1}{2}}$	$\dots$	$\rho_{n-1} \oplus \rho_1$
$\mathbb{R}$	ρ_0	ψ_0 (reducible)	ψ_1		$\psi_{\frac{n-1}{2}}$	$\psi_{\frac{n+1}{2}}$		ψ_{n-1}

$\Rightarrow$ In the extensions of $\rho_0, \psi_1, \dots, \psi_{\frac{n-1}{2}}$ we find $1 + 2 \cdot \frac{n-1}{2} = n$ distinct complex characters.

by the fact

$\Rightarrow$ for n odd, $\rho_0, \psi_1, \dots, \psi_{\frac{n-1}{2}}$ is a complete set of pairwise non-isomorphic real representations of G $\left(\# = 1 + \frac{n-1}{2} = \frac{n+1}{2}\right)$

For n even we have the following picture:

$\mathbb{C}$	ρ_0	$\rho_{n/2}$	$\rho_0 \oplus \rho_0$	$\rho_1 \oplus \rho_{n-1}$	$\dots$	$\rho_{\frac{n-2}{2}} \oplus \rho_{\frac{n+2}{2}}$	$\rho_{n/2} \oplus \rho_{n/2}$	$\rho_{\frac{n+2}{2}} \oplus \rho_{\frac{n-2}{2}}$	$\dots$	$\rho_{n-1} \oplus \rho_1$
$\mathbb{R}$	ρ_0	$\rho_{n/2}$	ψ_0	ψ_1		$\psi_{\frac{n-2}{2}}$	$\psi_{n/2}$	$\psi_{\frac{n+2}{2}}$		ψ_{n-1}
			reducible				reducible			

$\Rightarrow$ In the extensions of $\rho_0, \rho_{n/2}, \psi_1, \dots, \psi_{\frac{n-2}{2}}$ we find $2 + 2 \cdot \frac{n-2}{2} = n$ distinct complex characters.

$\Rightarrow$ for n even, $\rho_0, \rho_{n/2}, \psi_1, \dots, \psi_{\frac{n-2}{2}}$ is a complete set of pairwise non-isomorphic real representations of G $\left(\# \, 2 + \frac{n-2}{2} = \frac{n+2}{2}\right)$

Note: the (irreducible) real representations ψ_k decompose over $\mathbb{C}$, hence they are not absolutely irreducible. Recall from Lam (1991):

(7.5) Theorem. *Let R be a k-algebra (not necessarily finite-dimensional), and let M be a simple left R-module with $\dim_k M < \infty$. The following statements are equivalent:*

(1) $End({}_R M) = k$.

(2) *The map $R \to End(M_k)$ expressing the R-action on M is surjective.*

(3) *For any field extension $K \supseteq k$, M^K is a simple R^K-module.*

(4) *There exists an algebraically closed field $E \supseteq k$ such that M^E is a simple R^E-module.*

If one (and hence all) of these conditions holds, we say that M is an absolutely simple (or absolutely irreducible) R-module.

So, $End(\psi_k)$ must be bigger than $\mathbb{R}$. We can determine exactly what it is using:

(7.4) Lemma. *Let R be a k-algebra (not necessarily of finite dimension over k) and let M, N be left R-modules, with $\dim_k M < \infty$. Then the natural map*

$$\theta : (Hom_R(M,N))^K \longrightarrow Hom_{R^K}(M^K, N^K)$$

is an isomorphism of K-vector spaces.

So,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathrm{End}(\psi_k) \xrightarrow{\simeq} \mathrm{End}(\psi_k^{\mathbb{C}}) \simeq \mathrm{End}(\rho_k \oplus \overline{\rho}_k).$$

Since the ρ_k are all absolutely irreducible ($\mathbb{C}$ algebraically closed), we have $\mathrm{Hom}(\rho_k, \rho_l) = \delta_{kl} \cdot \mathbb{C}$, hence assuming $k \neq \frac{n}{2}$ if n even (ψ_k reducible anyways) we have $\rho_k \not\simeq \overline{\rho_k}$ and therefore

$$\dim_{\mathbb{C}} \mathrm{End}(\psi_k^{\mathbb{C}}) = 2 \Rightarrow \dim_{\mathbb{R}} \mathrm{End}(\psi_k) = 2.$$

What is the non-trivial endomorphism of ψ_k?

Recall that $\psi_k(g) = R_k$, so a linear map $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix A in standard basis is G-equivariant if and only if

$$A \cdot R_k = R_k \cdot A \qquad (\Leftrightarrow f(gv) = g f(v))$$

Now,

drop arguments for readability

$$\mathrm{I} \quad \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\cos - c\sin & b\cos - d\sin \\ a\sin + c\cos & b\sin + d\cos \end{pmatrix}$$

$$\mathrm{II} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} = \begin{pmatrix} a\cos + b\sin & -a\sin + b\cos \\ c\cos + d\sin & -c\sin + d\cos \end{pmatrix}$$

You can check that $\mathrm{I} = \mathrm{II}$ (assuming $k \neq \frac{n}{2}$ if n even $\Leftrightarrow \varphi_k \neq \pi \Leftrightarrow \sin \neq 0$) if and only if $a = d$ and $b = -c$. Hence,

$$\mathrm{End}(\psi_k) = \mathbb{R} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbb{R} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$