Symplectic singularities

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Introduction

Symplectic singularities are intriguing mathematical objects that were introduced by Beauville [1] in 2000. They are already interesting from a purely geometric perspective but an especially fascinating feature is that they naturally arise in representation theory (Lie theory) as well where they established a fruitful link between the commutative world of algebraic geometry and the noncommutative world of representation theory.

In this survey I would like to highlight some computational aspects that arise in the context of symplectic singularities. This is part of recent joint work [6] with Cédric Bonnafé. The computational approach allowed us to solve several mathematical problems which are probably hard (or impossible) to tackle by purely theoretical means. There are still many more questions in this area that may be attacked by computational methods and I invite everyone to join in.

Symplectic singularities

Throughout, we work over the complex numbers. I already want to note, however, that all the relevant objects can also be realized over some "sufficiently large" number field (a cyclotomic field actually) so that computations in computer algebra systems are possible. All our vector spaces will be finite-dimensional.

Symplectic vector spaces

A *symplectic form* on a vector space V is a bilinear form $\omega\colon V \times V \to \mathbb{C}$ which is *skew-symmetric*, i.e. $\omega(u,v) = -\omega(v,u)$, and which is *nondegenerate*, i.e. $\omega(u,v) = 0$ for all v implies $u = 0$. The *Gram matrix* of ω with respect to some basis $(v_i)_{i=1}^m$ of V is defined by

$$J_{ij} = \omega(v_i, v_j) \;, \tag{1}$$

so $\omega(v,u) = \mathbf{v} J \mathbf{u}^{\mathrm{T}}$, where $\mathbf{u}, \mathbf{v}$ denote the coordinate vectors of u, v in the basis. The matrix J is skew-symmetric, i.e. $J^T = -J$, and invertible. Conversely, any skew-symmetric invertible matrix defines a symplectic form on V via (1). Note that

$$\det(J) = \det(J^T) = \det(-J) = (-1)^n \det(J)$$

and this can only be true if n is *even*. Hence, a symplectic form only exists on an even-dimensional space.

The following will be our main example of a symplectic form. Let $\mathfrak{h}$ be any vector space (the symbol $\mathfrak{h}$ comes from the Lie-theoretic context where this example usually comes from). We denote by $\mathfrak{h}^*$ the dual vector space of $\mathfrak{h}$, i.e. the space of linear maps $\mathfrak{h} \to \mathbb{C}$. Then on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ there is a natural symplectic form defined by

$$\omega((v,f),(v',f')) = f'(v) - f(v') \;. \tag{2}$$

If $(y_i)_{i=1}^n$ is a basis of $\mathfrak{h}$ with dual basis $(x_i)_{i=1}^n$ then the Gram matrix of ω in the basis $\{y_1,\ldots,y_n,x_1,\ldots,x_n\}$ is

$$J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix} \;. \tag{3}$$

This example is in fact universal: a variant of the Gram–Schmidt process shows that after an appropriate change of basis the Gram matrix of any symplectic form is given by (3).

Poisson structure

Why care about symplectic structures? When considering the vector space $\mathfrak{h} \oplus \mathfrak{h}^*$ as an algebraic variety, its ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ of polynomial functions is by definition the symmetric algebra of the *dual* $\mathfrak{h}^* \oplus \mathfrak{h}$ of $\mathfrak{h} \oplus \mathfrak{h}^*$, and after our choice of basis this is simply the polynomial ring $A = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots y_n]$. For polynomials $f,g \in A$ define

$$\{f,g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} \right) \;. \tag{4}$$

In particular,

$$\{x_i,x_j\} = 0 = \{y_i,y_j\} \;, \quad \{x_i,y_j\} = \delta_{ij} \;. \tag{5}$$

You can check that $\{\cdot,\cdot\}$ is a *Lie bracket* on A that additionally satisfies the *Leibniz rule*

$$\{fg, h\} = \{f, h\}g + f\{g, h\} \, . \qquad (6)$$

Such a Lie bracket is called a *Poisson bracket* and A is called a *Poisson algebra*.

One can deduce the Poisson bracket also in a coordinate-free manner from a symplectic form (see [14, §1.11]). Poisson brackets are important in physics because they are the foundation of the Hamiltonian equations of motion.

Smooth symplectic varieties

In general, the configuration space of a mechanical system is not flat. Hence, one considers symplectic and Poisson structures on manifolds. Algebraists consider them on varieties.

Let X be a smooth variety. In each point $p \in X$ we have the tangent space T_pX. Since X is smooth, the dimension of T_pX is equal to the dimension of X as a variety. Intuitively, a symplectic form on X should be a "smoothly varying" family $(\omega_p)_{p \in X}$ of symplectic forms ω_p on T_pX. But this family should come from something global, so the correct definition of a *symplectic form* on X is a closed holomorphic 2-form ω on X such that the induced form ω_p on T_pX is nondegenerate for all $p \in X$. Note that a 2-form is alternating by definition so that the induced form ω_p on T_pX is indeed symplectic. As in the flat case, a symplectic form ω on X induces a Poisson bracket on $\mathbb{C}[X]$ (see [7, Theorem 1.2.7]). The closedness assumption on ω is needed to prove the Jacobi identity. A smooth variety equipped with a symplectic form is called a *smooth symplectic variety*.

A vector space with a symplectic form is a smooth symplectic variety. From the geometric perspective, our main example $\mathfrak{h} \oplus \mathfrak{h}^*$ is actually the cotangent bundle $T^*\mathfrak{h}$ of $\mathfrak{h}$. This example globalizes: the cotangent bundle T^*X of any smooth variety X is a smooth symplectic variety (see [7, Example 1.1.3]).

Symplectic varieties and symplectic singularities

If X has a singularity in a point p, then the dimension of T_pX is larger than the dimension of X. Since the smooth part X^{sm} of X is a non-empty open subset of X, the dimension of the tangent spaces is not constant on X and therefore we cannot consider a "smoothly varying" family of symplectic (and thus nondegenerate) forms on the tangent spaces as in the smooth case.

What is a good extension of the notion of smooth symplectic varieties to the singular world? Certainly, we would want X^{sm} to carry a symplectic form ω as before. It also makes sense to assume that X is normal. The extra ingredient making this into a good concept is due to Beauville [1]: we require that for any resolution $\pi \colon Y \to X$ of singularities (a proper birational map with Y smooth) the pullback of ω to $\pi^{-1}(X^{\mathrm{sm}})$ extends to a (possibly degenerate) holomorphic 2-form on all of Y. We then say that X is a *symplectic variety*. A singularity of a variety is said to be *symplectic* if it has an open neighborhood which is a symplectic variety. Symplectic singularities are rational Gorenstein (see [1, Proposition 1.3]).

The extra condition about the pullback of the form just needs be tested for one resolution. A result by Flenner [11] implies that it holds automatically if the singular locus is of codimension at least 4. Beauville adds: "We chose to impose it in all cases in order to get uniform results." Note that it is *not* assumed that the pullback extends to a nondegenerate (and thus symplectic) form. In this case Y is a smooth symplectic variety and π is called a *symplectic resolution*. In general, this does not exist (see the references in the next section).

There are two main classes of examples of symplectic singularities.

Example 1 First, we note that if G is a finite group of linear automorphisms of a vector space V, then the orbit space V/G has the structure of an algebraic variety with coordinate ring being the ring $\mathbb{C}[V]^G$ of G-invariant polynomial functions on V.

Now, suppose that V is symplectic with symplectic form ω. The *symplectic group* $\mathrm{Sp}(V) = \mathrm{Sp}(V, \omega)$ consists of the linear automorphisms g of V which leave ω invariant, i.e. $\omega(gv, gu) = \omega(v, u)$. If G is a finite subgroup of $\mathrm{Sp}(V)$, then V/G is a symplectic variety (see [1, Proposition 2.4]). The symplectic form on the smooth locus of V/G is induced from ω. This example globalizes: if X is a symplectic variety with symplectic form ω and G is a finite group of automorphisms of X leaving ω invariant, then X/G is a symplectic variety.

Note that $\mathrm{Sp}(V)$ is contained in the special linear group $\mathrm{SL}(V)$. If V is two-dimensional, both groups are equal and so the resulting symplectic singularities are precisely the Kleinian singularities, which are classified by Dynkin diagrams.

Example 2 Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $\mathcal{N}$ be the *nilpotent cone* of $\mathfrak{g}$, i.e. the set of all nilpotent elements of $\mathfrak{g}$. This is an irreducible variety (see [9, 8.1.3]). The adjoint group G of $\mathfrak{g}$ acts on $\mathcal{N}$ (see [8, §1.2]). An orbit of this action is called a *nilpotent orbit*. The normalization of the closure of a nilpotent orbit has symplectic singularities (see [1, §2.6]). The symplectic form on the smooth locus is a Kostant–Kirillov form.

Questions

There are many questions one can ask about symplectic singularities. Are there other examples? Can we classify them (up to isomorphism of analytic germs)? When does a symplectic resolution exist? What can we say about the birational geometry of a symplectic singularity in light of the minimal model program? How is the representation theory of the group or Lie algebra intertwined with the geometry?

This survey is not the right place to go into any of this. I recommend as starting points the surveys [12, 14]. These are rather old now, however, and there was much progress in the meantime. I therefore also recommend looking at the papers [3, 4] and the thesis [19].

Classifying symplectic singularities in general is problematic because by Example 1 we can always take a quotient by a finite group to produce new ones. Beauville therefore considers symplectic singularities with trivial *local fundamental group* to get rid of such examples. In a simple complex Lie algebra there is a unique non-zero minimal nilpotent orbit and its closure has an isolated symplectic singularity which in case $\mathfrak{g}$ is not of type C has trivial local fundamental group [1, Proposition 4.2]. Beauville asked in [1, §4.3] whether there are further examples. This was unanswered for 20 years.

Recently, a new *infinite* family of isolated symplectic singularities with trivial local fundamental group was discovered by the group Bellamy, Bonnafé, Fu, Juteau, Levy, and Sommers [5]. I will illustrate how one can arrive at the smallest of the new examples by the computational tools that I developed with Bonnafé in [6]. We first need to discuss the theoretical context in which they arise: Poisson deformations.

Poisson deformations

A method to study singularities—and to possibly create interesting new ones—is via *deformations*. When we have a symplectic singularity we would like to consider only deformations deforming the symplectic structure as well. This is problematic, however, since the symplectic form exists only on the smooth locus and not globally.

Remember Poisson brackets? They have a key advantage over symplectic forms, namely they exist *globally*. Recall that if X is a smooth symplectic variety, then the symplectic form on X induces a Poisson bracket on $\mathbb{C}[X]$. Hence, if X is a (not necessarily smooth) symplectic variety, then since X^{sm} carries a symplectic form, we have a Poisson bracket on $\mathbb{C}[X^{\mathrm{sm}}]$. Now, X is normal by definition, therefore $X \setminus X^{\mathrm{sm}}$ is of codimension ≥ 2 in X, and now the algebraic Hartog's theorem (see [13, Theorem 6.45]) implies that the restriction map $\mathbb{C}[X] \to \mathbb{C}[X^{\mathrm{sm}}]$ is an isomorphism so that the Poisson bracket extends uniquely from $\mathbb{C}[X^{\mathrm{sm}}]$ to $\mathbb{C}[X]$. This global Poisson bracket is easily obtained in the case of symplectic quotient singularities V/G as in Example 1: the Poisson bracket on $\mathbb{C}[V]$ is G-invariant and thus restricts to a Poisson bracket on the coordinate ring $\mathbb{C}[V]^G$ of V/G.

Now, let A be a Poisson algebra over $\mathbb{C}$, e.g. $A = \mathbb{C}[X]$ for a symplectic variety X. Let R be a $\mathbb{C}$-algebra. A *flat family of Poisson deformations* of A over R is a flat R-algebra $\tilde{A}$ equipped with an R-linear Poisson bracket such that $\tilde{A}/\mathfrak{m}_0\tilde{A} \simeq A$ as Poisson algebras, where $\mathfrak{m}_0$ is some fixed maximal ideal in R. Note that the Poisson bracket on $\tilde{A}$ induces a Poisson bracket on any quotient $\tilde{A}/\mathfrak{m}A$ by a maximal ideal $\mathfrak{m}$ of R so that we really get a family of Poisson deformations A and one of them is our original algebra A. Geometrically, $\mathrm{Spec}(\tilde{A})$ is a flat family of Poisson deformations of the *Poisson variety* $\mathrm{Spec}(A)$ over the *base* $\mathrm{Spec}(R)$.

Calogero–Moser spaces

How can we construct Poisson deformations of an affine symplectic variety X? Naively, we would like to take a presentation of the ring $\mathbb{C}[X]$ and then deform the relations appropriately. But this is problematic because we usually do not have a good understanding of X and we do not know a presentation of $\mathbb{C}[X]$ (see the last section on computations for an example illustrating this). For quotient singularities V/G, there is a beautiful solution due to Etingof and Ginzburg [10]. The Poisson deformations of V/G are called *Calogero–Moser spaces*. They have numerous applications in geometry, representation theory, and physics.

The basic idea

We start with the *skew group ring* $\mathbb{C}[V] \rtimes G$. This is simply the group ring of G over $\mathbb{C}[V]$, i.e. the free $\mathbb{C}[V]$-module with basis the elements of G, and multiplication defined by

$$(f_1 g_1)(f_2 g_2) = f_1 f_2^{g_1} g_1 g_2 \tag{7}$$

for $g_i \in G$ and $f_i \in \mathbb{C}[V]$. Here, f^g is the induced right action of $g \in G$ on polynomial functions $f \in \mathbb{C}[V]$. Note that $\mathbb{C}[V] \rtimes G$ is an *infinite-dimensional* $\mathbb{C}$-algebra and it is *noncommutative* if G acts nontrivially on V.

What has this algebra to do with V/G? It is an easy exercise to show that the *center* of $\mathbb{C}[V] \rtimes G$ is equal to $\mathbb{C}[V]^G$. We can therefore think of $\mathbb{C}[V] \rtimes G$ as the "noncommutative coordinate ring" of V/G. Etingof and Ginzburg [10] add: "[...] it is believed that the 'right' geometry of the G-action on X can be read off from the 'non-commutative algebraic geometry' of $\mathbb{C}[V] \rtimes G$."

The skew group ring has an easy presentation: since $\mathbb{C}[V]$ is just a polynomial ring, we only need the relations in G together with the relations (7) encoding the action of G on V. The idea is now to deform $\mathbb{C}[V] \rtimes G$ and take the center of the deformations to get deformations of V/G.

Symplectic reflection algebras

Etingof and Ginzburg [10] showed that this idea indeed works. They constructed a flat R-algebra $\tilde{H}$ over a polynomial ring $\tilde{R} = \mathbb{C}[T, C_1, \ldots, C_N]$ for a certain N depending on G such that center Z of $H = \tilde{H}/T\tilde{H}$ is a flat family of Poisson deformations of $\mathbb{C}[V]^G$ over $R = \tilde{R}/T\tilde{R}$. So, in particular, for any $c = (c_1, \ldots, c_N) \in \mathbb{C}^N$ the center Z_c of

$$H_c = H/\{C_1 - c_1, \ldots, C_N - c_N\}H$$

is a Poisson deformation of $\mathbb{C}[V]^G$. The associated variety X_c is called a *Calogero–Moser space*. It is an irreducible normal Gorenstein variety equipped with a Poisson bracket.

The algebra $\tilde{H}$ is called the *symplectic reflection algebra*. The name stems from the fact that the integer N

above is the number of conjugacy classes of *symplectic reflections* in G, i.e. elements whose fixed space is of codimension 2 in V. The symplectic reflections play a key role in the construction of $\tilde{H}$.

Just to give an idea, $\tilde{H}$ is defined as the quotient of $T(V) \rtimes G$, where $T(V)$ is the tensor algebra of V, by relations of the form

$$vw - wv = \kappa(v,w) \in RG \ , \tag{8}$$

for $v, w \in V$, where $\kappa(v,w)$ denotes some explicit element in the group ring RG which I am not going to specify any further here. When setting $T = 0$ and $C_i = 0$ for all i, this element is zero so that H_0 is equal to $\mathbb{C}[V] \rtimes G$ and therefore $Z_0 = \mathbb{C}[V]^G$.

The Poisson bracket on Z_c comes from a commutator in $\tilde{H}$, which explains why we need the additional parameter T as well. It is an amazing fact the Calogero–Moser spaces yield *all* the Poisson deformations of V/G (see [2]). From an algebraic point of view, the general setting of symplectic reflection algebras is that of *filtered deformations* (see [16]).

The fact that H is a family of deformations of $\mathbb{C}[V] \rtimes G$ implies that as a vector space H_c is isomorphic to $\mathbb{C}[V] \rtimes G$. This fact is called the *Poincaré–Birkhoff–Witt theorem*. In particular, H_c has a nice basis consisting of elements of the form fg with $f \in \mathbb{C}[V]$ and $g \in G$. Such a basis is called a *PBW basis*.

Computational approach

Calogero–Moser spaces are again (not necessarily smooth) symplectic varieties (see [14, Proposition 4.5]). What kind of symplectic singularities do they have? The new symplectic singularities discovered in [5] are indeed singularities of Calogero–Moser spaces, so this seems to be an interesting question. It would be exciting if we could construct and study Calogero–Moser spaces in the computer to do experiments. This is precisely the topic of my recent work with Bonnafé [6]. I want to note that the study of singularities is just one of the many facets of our computational approach.

Symplectic singularities associated to complex reflection groups

I will from now on restrict to a special (but very important) case of symplectic quotient singularities. Our algorithms from [6] are adapted to this setting. Let $\mathfrak{h}$ be a vector space. Recall from (2) that $T^*\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}^*$ carries a symplectic form. If W is a finite group of linear automorphisms of $\mathfrak{h}$, then W naturally acts on $\mathfrak{h}^*$ and thus on $T^*\mathfrak{h}$ as well. This action clearly leaves the symplectic form invariant, hence $T^*\mathfrak{h}/W$ is a symplectic variety.

We assume that W is a *complex reflection group*, i.e. W is generated by elements s whose fixed space is a hyperplane in $\mathfrak{h}$. These groups naturally arise in algebraic geometry: it is a classical fact that $\mathfrak{h}/W$ is smooth if and only if W is a complex reflection group. Complex reflection groups have been classified by Shephard and Todd [20]. The symplectic reflection algebra for this special case has a special name: the *rational Cherednik algebra*. The symplectic reflections are in one-to-one correspondence with the complex reflections.

Backbone of the computations

Recall that the Calogero–Moser space is the center of the rational Cherednik algebra. So, in order to construct this variety in the computer, we first need to be able to compute in the rational Cherednik algebra itself.

I need to clarify what I mean by "compute". Recall that the rational Cherednik algebra H_c is as a vector space isomorphic to $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W$ and it therefore has a nice basis (PBW basis). Now, by "compute" I mean that we can *rewrite* the product of two elements again in a PBW basis so that we can test for equality etc. Experts may think of Groebner basis computations but Groebner theory is actually not needed to work with these algebras because the rewrite relations are straightforward from the defining relations (8).

I achieved this a while ago (motivated by applications in representation theory) in form of a theoretical algorithm [21] together with an implementation in my software package CHAMP (Cherednik Algebra Magma Package) [22] based on the computer algebra system MAGMA [17]. The reasons for choosing MAGMA was that the computations require a fast computer algebra system that can do group theory, representation theory, algebraic geometry, number theory, etc. In the meantime, the new computer algebra system OSCAR [18] was developed having similar characteristics but also having the advantage of being open source.

Computing the center

Being able to compute in rational Cherednik algebras is nice but how do we get the center? This is the starting point of my work with Bonnafé [6]. The key observation is that the natural "truncation" map

$$\mathrm{Trunc}\colon H \to R[\mathfrak{h} \oplus \mathfrak{h}^*] \tag{9}$$

sending $h \in H$ to the coefficient of $1 \in W$ in a PBW basis restricts to an isomorphism between the center Z of H and the invariant ring $R[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. The latter can be handled with computational invariant theory and we found a way to compute the inverse of Trunc using an inductive deformation procedure. In this way we can deform a system of fundamental invariants for the action of W on $T^*\mathfrak{h}$ to a system of generators of Z as an R-algebra.

Similarly, we found a way to deform relations between the fundamental invariants to relations between the generators of Z, ultimately giving a presentation of Z. After specialization in a point $c \in \mathbb{C}^N$, this gives a presentation of Z_c. We can also explicitly compute the Poisson bracket on Z_c. This again is based on computations in the rational Cherednik algebra itself.

Instead of going into further details, I will illustrate our computational approach and its power in an explicit example.

An example

We consider $T^*\mathfrak{h}/W$ where W is the dihedral group of order $2d$ acting in a reflection representation on $\mathfrak{h} = \mathbb{C}^2$. Explicitly, W is generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \zeta \\ \zeta^{-1} & 0 \end{pmatrix} ,$$

where ζ is a primitive 5th root of unity. This group is created in MAGMA/CHAMP as follows:

```
> W := ShephardTodd(5,5,2); W;
MatrixGroup(2, Cyclotomic Field of order 5 and
    degree 4)
Generators:
    [0 1]
    [1 0]

    [                                0
                                zeta_5]
    [-zeta_5^3 - zeta_5^2 - zeta_5 - 1
                                     0]
```

To get an idea about the complexity of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ in this seemingly simple case, we compute a system of fundamental invariants and relations between them:

```
> SymplecticDoublingFundamentalInvariants(W);
[
    y1*y2,
    y1*x1 + y2*x2,
    x1*x2,
    y1^5 + y2^5,
    y1^4*x2 + y2^4*x1,
    y1^3*x2^2 + y2^3*x1^2,
    y1^2*x2^3 + y2^2*x1^3,
    y1*x2^4 + y2*x1^4,
    x1^5 + x2^5
]
> SymplecticDoublingInvariantRingPresentation(W);
Ideal of Polynomial ring of rank 9 over Cyclotomic
     Field of order 5 and degree 4
Order: Lexicographical
Variables: z1, z2, z3, z4, z5, z6, z7, z8, z9
Basis:
[
    z1*z6 - z2*z5 + z3*z4,
    z1*z7 - z2*z6 + z3*z5,
    z1*z8 - z2*z7 + z3*z6,
    z1*z9 - z2*z8 + z3*z7,
    -4*z1^2*z2*z3^2 + 5*z1*z2^3*z3 - z2^5 + z4*z9
        - z6*z7,
    4*z1^4*z3 - z1^3*z2^2 + z4*z6 - z5^2,
    4*z1^3*z2*z3 - z1^2*z2^3 + z4*z7 - z5*z6,
    -4*z1^3*z3^2 + 5*z1^2*z2^2*z3 - z1*z2^4 + z4*
        z8 - z5*z7,
    -4*z1^2*z2*z3^2 + z1*z2^3*z3 - z5*z8 + z6*z7,
    4*z1^2*z2^2*z3 - z1*z2^4 + z4*z8 - z6^2,
    -4*z1^2*z3^3 + 5*z1*z2^2*z3^2 - z2^4*z3 + z5*
        z9 - z6*z8,
    4*z1*z2^2*z3^2 - z2^4*z3 + z5*z9 - z7^2,
    4*z1*z2*z3^3 - z2^3*z3^2 + z6*z9 - z7*z8,
    4*z1*z3^4 - z2^2*z3^3 + z7*z9 - z8^2
]
```

Here, "symplectic doubling" refers to considering the action of W on $T^*\mathfrak{h}$.

Next, we create the rational Cherednik algebra H_0:

```
> H := RationalCherednikAlgebra(W,0 : Type:="BR-K"
    );
```

The "Type" argument is about the particular form of parameters used. One can now compute in this algebra using a PBW basis. We refrain from such computations here and straightaway compute a presentation of Z:

```
> time CenterPresentation(H):
[
    z1*z6 + -1*z2*z5 + z3*z4,
    z1*z7 + -1*z2*z6 + z3*z5,
    z1*z8 + -1*z2*z7 + z3*z6,
    z1*z9 + -1*z2*z8 + z3*z7,
    -4*z1^2*z2*z3^2 + 5*z1*z2^3*z3 + -100*K1_1^2*
        z1*z2*z3 + -1*z2^5 +
        100*K1_1^2*z2^3 + z4*z9 + -1*z6*z7,
    4*z1^4*z3 + -1*z1^3*z2^2 + 100*K1_1^2*z1^3 +
        z4*z6 + -1*z5^2,
    4*z1^3*z2*z3 + -1*z1^2*z2^3 + 100*K1_1^2*z1^2*
        z2 + z4*z7 + -1*z5*z6,
    -4*z1^3*z3^2 + 5*z1^2*z2^2*z3 + -100*K1_1^2*z1
        ^2*z3 + -1*z1*z2^4 +
        100*K1_1^2*z1*z2^2 + z4*z8 + -1*z5*z7,
    -4*z1^2*z2*z3^2 + z1*z2^3*z3 + -100*K1_1^2*z1*
        z2*z3 + -1*z5*z8 + z6*z7,
    4*z1^2*z2^2*z3 + -1*z1*z2^4 + 100*K1_1^2*z1*z2
        ^2 + z4*z8 + -1*z6^2,
    -4*z1^2*z3^3 + 5*z1*z2^2*z3^2 + -100*K1_1^2*z1
        *z3^2 + -1*z2^4*z3 +
        100*K1_1^2*z2^2*z3 + z5*z9 + -1*z6*z8,
    4*z1*z2^2*z3^2 + -1*z2^4*z3 + 100*K1_1^2*z2^2*
        z3 + z5*z9 + -1*z7^2,
    4*z1*z2*z3^3 + -1*z2^3*z3^2 + 100*K1_1^2*z2*z3
        ^2 + z6*z9 + -1*z7*z8,
    4*z1*z3^4 + -1*z2^2*z3^3 + 100*K1_1^2*z3^3 +
        z7*z9 + -1*z8^2
]
Time: 1.600
```

The computation took only 1.6 seconds. If you look closely, you see there is just one parameter in this case (denoted `K1_1`). We want to study the Calogero–Moser space when we specialize the parameter to 1:

```
> X := CalogeroMoserSpace(H, [1]);
```

This command creates the Calogero–Moser space X as an affine scheme over $\mathbb{Q}(\zeta)$ in MAGMA. There is no magic scheme theory happening here: this only sets up a geometric context for an ideal in a polynomial ring so that we can conveniently ask geometric questions. For example, we compute the singular locus of X:

```
> time Xsing := SingularSubscheme(X);
Time: 19.020
```

The output is *horrendous*! We make it simpler by computing the reduced subscheme structure (taking the radical):

```
> time Xsing := ReducedSubscheme(Xsing);
Time: 651.340
```

In comparison to all computations so far, this one takes a lot of time. The output looks much simpler but we can make it even simpler by computing a minimal basis:

```
> time MinimalBasis(Xsing);
[
    z9,
    z8,
    z7,
    z6,
    z5,
    z4,
    z3,
    z2,
    z1
]
Time: 0.000
```

So, surprise, the Calogero–Moser space X has an isolated singularity at the origin! We know from theory that it is a symplectic singularity. Is it a known one, i.e. does it belong to Example 1 or Example 2?

Beauville [1] characterized minimal nilpotent orbit singularities (recall that these have outside of type C trivial local fundamental group) as those for which their projective tangent cone is *smooth*.

Let us test this condition. We denote the point of X corresponding to the origin by o. First, we compute the tangent cone:

```
> o := X![0,0,0,0,0,0,0,0,0];
> cone := TangentCone(X, o);
> cone;
Scheme over Rational Field defined by
z7*z9 - z8^2,
z6*z9 - z7*z8,
z5*z9 - z7^2,
z4*z9 - z6*z7,
z1*z9 - z2*z8 + z3*z7,
z6*z8 - z7^2,
z5*z8 - z6*z7,
z4*z8 - z6^2,
z1*z8 - z2*z7 + z3*z6,
z5*z7 - z6^2,
z4*z7 - z5*z6,
z1*z7 - z2*z6 + z3*z5,
z4*z6 - z5^2,
z1*z6 - z2*z5 + z3*z4,
z1*z5^2 - z2*z4*z5 + z3*z4^2
```

Next, we take projective closure of the tangent cone:

```
> projcone := Scheme(Proj(CoordinateRing(
    AmbientSpace(X))), MinimalBasis(cone));
> projcone;
Scheme over Rational Field defined by
z7*z9 - z8^2,
z6*z9 - z7*z8,
z6*z8 - z7^2,
z5*z9 - z7^2,
z5*z8 - z6*z7,
z5*z7 - z6^2,
z4*z9 - z6*z7,
z4*z8 - z6^2,
z4*z7 - z5*z6,
z4*z6 - z5^2,
z1*z9 - z2*z8 + z3*z7,
z1*z8 - z2*z7 + z3*z6,
z1*z7 - z2*z6 + z3*z5,
z1*z6 - z2*z5 + z3*z4
```

Now, we check whether the projective tangent cone is singular:

```
> time IsSingular(projcone);
true
Time: 119.580
```

The conclusion is that $(X, 0)$ is *not* a minimal nilpotent orbit singularity!

We do not yet have a computational approach to the local fundamental group. In [5] it was proven that $(X, 0)$ has trivial local fundamental group. Hence, this is a new example of a symplectic singularity! This indeed works for all dihedral groups of order $2d$ with $d \geq 5$, giving an *infinite* family of new examples.

Closing remarks

By similar computations we have shown in [6] that for the *exceptional* complex reflection group of type G_4 there is a parameter c such that the Calogero–Moser space X_c has an isolated symplectic singularity and this singularity is equivalent to the minimal nilpotent orbit singularity for the Lie algebra of type $\mathfrak{sl}_3$. I am convinced that this result cannot be derived by purely theoretical arguments yet and I do not understand why $\mathfrak{sl}_3$ arises in the context of G_4. There is still a lot to discover.

References

[1] Beauville, A. (2000). Symplectic singularities. *Invent. Math., 139(3)*, 541–549.

[2] Bellamy, G. (2016). Counting resolutions of symplectic quotient singularities. *Compos. Math., 152*(1), 99–114.

[3] Bellamy, G., & Craw, A. (2020). Birational geometry of symplectic quotient singularities. *Invent. Math., 222*(2), 399–468.

[4] Bellamy, G., Schmitt, J., & Thiel, U. (2022). Towards the classification of symplectic linear quotient singularities admitting a symplectic resolution. *Math. Z., 300*(1), 661–681.

[5] Bellamy, G., Bonnafé, C., Fu, B., Juteau, D., Levy, P., & Sommers, E. (2023). A new family of isolated symplectic singularities with trivial local fundamental group. *Proc. Lond. Math. Soc. (3), 126*(5), 1496–1521.

[6] Bonnafé, C. & Thiel, U. (2023). Computational aspects of Calogero–Moser spaces. *Selecta Math. (N.S.)*, to appear.

[7] Chriss, N., & Ginzburg, V. (1997). *Representation theory and complex geometry*. Birkhäuser Boston.

[8] Collingwood, D. H., & McGovern, W. M. (1993). *Nilpotent orbits in semisimple Lie algebras* Van Nostrand Reinhold Co.

[9] Dixmier, J. (1996). *Enveloping algebras*. Graduate Studies in Mathematics, Vol. 11. American Mathematical Society.

[10] Etingof, P., & Ginzburg, V. (2002). Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math., 147*(2), 243–348.

[11] Flenner, H. (1988). Extendability of differential forms on nonisolated singularities. *Invent. Math., 94*(2), 317–326.

[12] Fu, B. (2006). A survey on symplectic singularities and symplectic resolutions. *Ann. Math. Blaise Pascal, 13*(2), 209–236.

[13] Görtz, U., & Wedhorn, T. (2010). *Algebraic geometry I.* Vieweg + Teubner.

[14] Gordon, I. G. (2008). Symplectic reflection algebras. In: *Trends in representation theory of algebras and related topics* (pp. 285–347). Eur. Math. Soc.

[15] Kaledin, D. (2006). Symplectic singularities from the Poisson point of view. *J. Reine Angew. Math., 600*, 135–156.

[16] Losev, I. (2012). *Lectures on symplectic reflection algebras.* https://web.northeastern.edu/iloseu/SRA/SRA.html

[17] Magma Computational Algebra System. http://magma.maths.usyd.edu.au/magma/

[18] OSCAR Computer Algebra System. https://www.oscar-system.org/

[19] Schmitt, J. (2023). *On $\mathbb{Q}$-factorial terminalizations of symplectic linear quotient singularities* PhD thesis, University of Kaiserslautern-Landau (RPTU). https://kluedo.ub.rptu.de/frontdoor/index/index/docId/7351

[20] Shephard, G. C., & Todd, J. A. (1954). Finite unitary reflection groups. *Canad. J. Math., 6*, 274–304.

[21] Thiel, U. (2015). Champ: a Cherednik algebra Magma package. *LMS J. Comput. Math., 18*(1), 266–307.

[22] Thiel, U. CHAMP. https://github.com/ulthiel/Champ/

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