AN ALGORITHM FOR COMPUTING THE CENTER OF A FUSION CATEGORY

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Abstract. We present an algorithm for explicitly computing the simple objects of the categorical (Drinfeld) center of a fusion category. Our approach is based on decomposing the images of simple objects under the induction functor from the category to its center. We have implemented this algorithm in a general-purpose software framework for tensor categories that we develop based on the high-performance programming language Julia and the open-source computer algebra system OSCAR. While the required computations are still too heavy to investigate established examples whose center is not yet known up to equivalence, our algorithm also works over not necessarily algebraically closed fields and this yields new explicit examples of non-split modular categories.

1. Introduction

Algorithmic techniques in group and representation theory have a long and successful history. Two, of the many, highlights are

and they are supported by many computer algebra systems. The algorithmic side of the categorical (categorified) level involving tensor categories and categorical representations, however, is still mostly unexplored and not supported. Generally, the goal should be to find algorithmic approaches to categorical constructions in this field that express the result in terms of the original input data, avoiding anything non-constructive like abstract equivalences. Our first main objective was to achieve this goal for the construction of the categorical center of a fusion category.

To initiate an investigation, we have begun developing a general-purpose open-source software framework `TensorCategories.jl` [2] to constructively work with structures in this realm.

Let $\mathcal{C}$ be a monoidal category with associators

$$a_{X,Y,Z} \colon (X \otimes Y) \otimes Z \xrightarrow{\simeq} X \otimes (Y \otimes Z) \,. \tag{1.1}$$

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Throughout, we will use conventions as in the standard reference [1]. Note that by Mac Lane's strictness theorem [1, Theorem 2.8.5], $\mathcal{C}$ is monoidally equivalent to a strict monoidal category, i.e. where all associators are the identity. However, due to our general goal we do not want to assume that $\mathcal{C}$ is strict. There is also a more fundamental reason to not assume this here. Namely, to encode a category in the computer we usually need to "discretize" it, meaning we need to choose a skeleton.

[1, Remark 2.8.7], comment CAP somewhat

Definition 1.1. A *half-braiding* for an object $X \in \mathcal{C}$ is a natural isomorphism

$$\gamma_X = \{\gamma_X(Y)\colon X \otimes Y \xrightarrow{\simeq} Y \otimes X \mid Y \in \mathcal{C}\}$$

such that

$$\begin{array}{ccccc} & (Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z) & \\ {\scriptstyle \gamma_X(Y)\otimes \mathrm{id}_Z} \nearrow & & & & \searrow {\scriptstyle \mathrm{id}_Y \otimes \gamma_X(Z)} \\ (X \otimes Y) \otimes Z & & & & Y \otimes (Z \otimes X) \\ {\scriptstyle a_{X,Y,Z}} \searrow & & & & \nearrow {\scriptstyle a_{Y,Z,X}} \\ & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_X(Y\otimes Z)} & (Y \otimes Z) \otimes X & \end{array}$$

commutes for all $Y, Z \in \mathcal{C}$ and $\gamma_1 = \mathrm{id}_X$.

Definition 1.2. The categorical *center* $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is given by the following data:

- Objects are tuples (X, γ_X) where $X \in \mathcal{C}$ and γ_X is a half-braiding.
- The sets $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \gamma_X), (Y, \gamma_Y))$ are given by morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that

$$\begin{array}{ccc} X \otimes Z & \xrightarrow{f \otimes \mathrm{id}_Z} & Y \otimes Z \\ \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma_Y} \\ Z \otimes X & \xrightarrow{\mathrm{id}_Z \otimes f} & Z \otimes Y \end{array} \tag{1.2}$$

  commutes for all $Z \in \mathcal{C}$.

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2. An algorithm based on the induction functor

From now on we assume $\mathcal{C}$ to be a spherical fusion category over an any field $\Bbbk$ with simple objects $X_1, ..., X_n$ such that $\dim \mathcal{C} \neq 0$ and $\mathrm{End}(X_i) = \Bbbk$ for all i. The last condition is usually known as $\mathcal{C}$ beeing *split semisimple*. Note that we explicitly do not require $\Bbbk$

to be algebraically closed. This is useful, since even though it is possible to work with algebraic numbers in the computer it is comparatively slow. Thus it is useful to work over any fixed field.

We recall some useful results.

Theorem 2.1 ([4, Theorem 1.2.]). *$\mathcal{Z}(\mathcal{C})$ is again a spherical fusion category over $\Bbbk$. Furthermore $\mathcal{Z}(\mathcal{C})$ is modular if it is split.*

Remark 2.2. The original statement deals only with the case where $\Bbbk$ is algebraically closed, but the proof does not rely on the algebraic closedness of $\Bbbk$ and thus transfers directly.

Lemma 2.3 ([4, Lemma 3.3]). *Let $\mathcal{C}$ be a fusion category with simple objects $\{X_i\}$. Let $Z \in \mathcal{C}$. There is a bijection between half-braidings for Z and families of morphisms $\{\gamma_Z(X_i) \in \mathrm{Hom}(Z \otimes X_i, X_i \otimes Z)\}$ such that for all i, j, k and $t \in \mathrm{Hom}(X_k, X_i \otimes X_j)$ the diagram*

$$\begin{array}{ccccc}
Z \otimes X_k & \xrightarrow{\gamma_Z(X_k)} & X_k \otimes Z & \xrightarrow{t \otimes \mathrm{id}_Z} & (X_i \otimes X_j) \otimes Z \\
\Big\downarrow{\scriptstyle \mathrm{id}_Z \otimes t} & & & & \Big\downarrow{\scriptstyle a_{X_i,X_j,Z}} \\
Z \otimes (X_i \otimes X_j) & & & & X_i \otimes (X_j \otimes Z) \\
\Big\downarrow{\scriptstyle a^{-1}_{Z,X_i,X_j}} & & & & \Big\uparrow{\scriptstyle \mathrm{id}_{X_i} \otimes \gamma_Z(X_j)} \\
(Z \otimes X_i) \otimes X_j & \xrightarrow{\gamma_Z(X_i) \otimes \mathrm{id}_{X_j}} & (X_i \otimes Z) \otimes X_j & \xrightarrow{a_{X_i,Z,X_j}} & X_i \otimes (Z \otimes X_j)
\end{array}$$

commutes and $\gamma_Z(\mathbb{1}) = \mathrm{id}_Z$.

Lemma 2.3 allows us to talk about half-braidings in discrete manner. All characterising information of a half-braiding is encoded in the isomorphisms $\gamma_Z(X_i)$ for simple objects X_i. From those all half-braidings can be easily build up, which allows also to obtain the braiding for $\mathcal{Z}(\mathcal{C})$. Moreover we are able to check in finite time whether a given set of half-braiding morphisms is indeed a half-braiding.

Now let X_i, X_j, X_k be simple and $t \in \mathrm{Hom}(X_k, X_i \otimes X_j)$. Then the lemma states an equation

$$\phi(\gamma_Z(X_k), t) = \psi(\gamma_Z(X_i), \gamma_Z(X_j), t)$$

where

$$\begin{aligned}
\phi(\gamma_Z(X_k), t) &= a_{X_i,X_j,X_k} \circ (t \otimes \mathrm{id}_Z) \circ \gamma_Z(X_k) \\
\psi(\gamma_Z(X_i), \gamma_Z(X_j), t) &= (\mathrm{id}_{X_i} \otimes \gamma_Z(X_j)) \circ a_{X_i,Z,X_j} \circ (\gamma_Z X_i \otimes \mathrm{id}_{X_j}) \circ \\
&\quad a^{-1}_{Z,X_i,X_j} \circ (\mathrm{id}_Z \otimes t).
\end{aligned}$$

Clearly ϕ and ψ are linear in t as well as in $\gamma_Z(X_i), \gamma_Z(X_j)$ and $\gamma_Z(X_k)$ respectively. Thus whenever the equation holds for a basis of $\mathrm{Hom}(X_k, X_i \otimes X_j)$ it holds for all t.

After choosing a basis $f_1, ..., f_r$ for $\mathrm{Hom}(Z \otimes X_k, X_i \otimes (Z \otimes X_j))$ and bases $g_1^l, ..., g_{r_l}^l$ for $\mathrm{Hom}(Z \otimes X_l, X_l \otimes Z)$ we can replace $\gamma_Z(X_l)$ with

$$\gamma_Z(X_l) = a_1^l g_1^l + \cdots + a_{r_l}^l g_{r_l}^l$$

and set up a system of quadratic equations by comparing coefficients.Using algebraic solvers like msolve[1] this yields an approach to finding simples in the center. Although the ideals generated by this system of equations often have positive dimension and thus an inifinite set of solutions. So first of all much computational work is required to make this happen since solving quadratic systems is very slow making the approach from this only useful in cases where other approches fail, e.g. if the category is not spherical.

Example 2.4. Consider the category $\mathcal{C} = \mathrm{Vec}_{\mathbb{Q}}(S_3)$. To use the approach above we need to examine candidates for central objects and then solve for the half-braidings. Let $\mathcal{K}_0(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$. Then every object in the image of the forgetful functor is mapped to an element of the center of $\mathcal{K}_0(\mathcal{C})$ under the projection. I.e. only objects corresponding to the center of $\mathcal{K}_0(\mathcal{C})$ need to be considered. For $\mathcal{C}$ this is generated by $\delta_{()}, \delta_{(12)} + \delta_{(13)} + \delta_{(23)}$ and $\delta_{(123)} + \delta_{(132)}$. When no structural properties are used to reduce the number of variables and equations this leads to already rather big ideals. In the case of $\delta_{(12)} + \delta_{(13)} + \delta_{(23)}$ this means $6 \cdot 3 = 18$ indeterminates in 108 equations.

[1]https://msolve.lip6.fr

$$\begin{aligned}\langle -x_1^2+x_1,&-x_2^2+x_2,-x_3^2+x_3,x_1-x_4^2,x_2-x_5x_6,x_3-x_5x_6,x_1-x_7x_9,x_2-x_8^2,\\
&x_3-x_7x_9,x_1-x_{10}x_{14},x_2-x_{11}x_{15},x_3-x_{12}x_{13},x_1-x_{12}x_{13},x_2-x_{10}x_{14},\\
&x_3-x_{11}x_{15},x_1-x_{16}x_{17},x_2-x_{16}x_{17},x_3-x_{18}^2,-x_1x_4+x_4,-x_2x_5+x_5,\\
&-x_3x_6+x_6,-x_3x_5+x_5,-x_2x_6+x_6,x_4-x_7x_{12},x_5-x_8x_{11},x_6-x_9x_{10},\\
&x_4-x_{10}x_{17},x_5-x_{11}x_{18},x_6-x_{12}x_{16},x_4-x_9x_{13},x_5-x_7x_{14},x_6-x_8x_{15},\\
&x_4-x_{14}x_{16},x_5-x_{13}x_{17},x_6-x_{15}x_{18},-x_1x_7+x_7,-x_2x_8+x_8,-x_3x_9+x_9,\\
&-x_4x_{13}+x_7,-x_5x_{15}+x_8,-x_6x_{14}+x_9,-x_3x_7+x_7,-x_1x_9+x_9,\\
&-x_5x_{10}+x_7,-x_6x_{11}+x_8,-x_4x_{12}+x_9,x_7-x_{13}x_{18},x_8-x_{14}x_{16},x_9-x_{15}x_{17},\\
&x_7-x_{11}x_{16},x_8-x_{10}x_{17},x_9-x_{12}x_{18},-x_1x_{10}+x_{10},-x_2x_{11}+x_{11},\\
&-x_3x_{12}+x_{12},-x_4x_{16}+x_{10},-x_5x_{18}+x_{11},-x_6x_{17}+x_{12},-x_6x_7+x_{10},\\
&-x_5x_8+x_{11},-x_4x_9+x_{12},-x_2x_{10}+x_{10},-x_3x_{11}+x_{11},-x_1x_{12}+x_{12},\\
&x_{10}-x_{13}x_{15},x_{11}-x_{13}x_{14},x_{12}-x_{14}x_{15},-x_8x_{16}+x_{10},-x_7x_{17}+x_{11},\\
&-x_9x_{18}+x_{12},-x_1x_{13}+x_{13},-x_2x_{14}+x_{14},-x_3x_{15}+x_{15},-x_4x_7+x_{13},\\
&-x_5x_9+x_{14},-x_6x_8+x_{15},-x_7x_{18}+x_{13},-x_8x_{17}+x_{14},-x_9x_{16}+x_{15},\\
&-x_{10}x_{11}+x_{13},-x_{11}x_{12}+x_{14},-x_{10}x_{12}+x_{15},-x_3x_{13}+x_{13},-x_1x_{14}+x_{14},\\
&-x_2x_{15}+x_{15},-x_5x_{16}+x_{13},-x_4x_{17}+x_{14},-x_6x_{18}+x_{15},-x_1x_{16}+x_{16},\\
&-x_2x_{17}+x_{17},-x_3x_{18}+x_{18},-x_4x_{10}+x_{16},-x_5x_{12}+x_{17},-x_6x_{11}+x_{18},\\
&-x_7x_{15}+x_{16},-x_8x_{14}+x_{17},-x_9x_{13}+x_{18},-x_8x_{10}+x_{16},-x_9x_{11}+x_{17},\\
&-x_7x_{12}+x_{18},-x_6x_{13}+x_{16},-x_4x_{14}+x_{17},-x_5x_{15}+x_{18},-x_2x_{16}+x_{16},\\
&-x_1x_{17}+x_{17},x_1-1,x_2-1,x_3-1\rangle\end{aligned}$$

As the number of simple objects and the multiplicity of the fusion coefficients goes up the number of variables and equations increases drastically and solving the resulting quadratic systems becomes practically impossible. Especially if the coefficients are not in $\mathbb{Q}$.

Since in most literature fusion categories are assumed to be strict and we explictly want to work with non-strict categories we will reformulate some results from [3].

Definition 2.5. Let $S, T, W \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(S, W \otimes T)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(S^* \otimes W, T^*)$. Let ψ be the spherical structure. There is a non-degenerate pairing

$$(f,g) = \mathrm{Tr}\left(\begin{array}{ccccc} W & \xrightarrow{\mathrm{coev}_S \otimes \mathrm{id}_W} & (S \otimes S^*) \otimes W & \xrightarrow{a_{S,S^*,W}} & S \otimes (S^* \otimes W) \\ & & & & \downarrow{\scriptstyle f \otimes g} \\ & & & & (W \otimes T) \otimes T^* \\ & & & & \downarrow{\scriptstyle a_{W,T,T^*}} \\ W & \xleftarrow[\mathrm{id}_W \otimes \mathrm{ev}_{T^*}]{} & W \otimes (T^{**} \otimes T^*) & \xleftarrow[\mathrm{id}_W \otimes (\psi_T \otimes \mathrm{id}_{T^*})]{} & W \otimes (T \otimes T^*) \end{array}\right) \tag{2.1}$$

Remark 2.6. The pairing from definition 2.5 is precisely the pairing defined in [3, Eq. 1.6] applied after transforming via the natural isomorphism $\hom(S^* \otimes W, T^*) \cong \hom(S^*, T^* \otimes W^*)$.

The following result is a rephrased version of [3, Theorem 2.3.].

Theorem 2.7. *Let $F\colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor and $I\colon \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ the (left) adjoint of F. Then for $X \in \mathcal{C}$ we have*

$$F(I(X)) = \bigoplus_{i=1}^{n} (X_i \otimes X) \otimes X_i^*.$$

The half-braiding for $Z \in \mathcal{C}$ is given by $\gamma(Z) = d^{-1} \circ (\gamma(Z)_{i,j}) \circ d'$ with

$$\gamma_{i,j}\colon ((X_i \otimes X) \otimes X_i^*) \otimes Z \to Z \otimes ((X_j \otimes X) \otimes X_j^*)$$

and d, d' the distributivity isomorphisms. Let B, B' be dual bases of $\mathrm{Hom}(X_i, Z \otimes X_j)$, *respectively* $\mathrm{Hom}(X_i^* \otimes Z, X_j^*)$, *with respect to the pairing 2.5. We then have*

$$\gamma(Z)_{i,j} = \dim X_i \sum_{f \in B, g \in B'} \begin{array}{ccc} ((X_i \otimes X) \otimes X_i^*) \otimes Z & \xrightarrow{a_{X_i \otimes X, X_i^*, Z}} & (X_i \otimes X) \otimes (X_i^* \otimes Z) \\ & & \downarrow{\scriptstyle (f \otimes \mathrm{id}_X) \otimes g} \\ & & ((Z \otimes X_j) \otimes X) \otimes X_j^* \\ & & \downarrow{\scriptstyle a_{Z, X_j, X} \otimes \mathrm{id}_{X_j^*}} \\ Z \otimes ((X_j \otimes X) \otimes X_j^*) & \xleftarrow[a_{Z, X_j \otimes X, X_j^*}]{} & (Z \otimes (X_j \otimes X)) \otimes X_j^* \end{array}.$$

We call the functor I the induction.

2.1. **Obtaining Morphisms in $\mathcal{Z}(\mathcal{C})$.** There are multiple ways to obtain morphisms in in the center of of a fusion category $\mathcal{C}$. The first and most straight forward approch is to solve for the condition 1.2. If $f \in \operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}(X,Y) \subset \operatorname{Hom}_{\mathcal{C}}(F(X),F(Y))$ we can write

$$f = a_1 g_1 + \cdots + a_n g_n$$

where $g_1, ..., g_n$ is a basis of $\operatorname{Hom}_{\mathcal{C}}(F(X),F(Y))$. Condition 1.2 yields now a set of equations

$$\gamma_Y(Z) \circ f \otimes \mathrm{id}_Z = \mathrm{id}_Z \otimes f \circ \gamma_X(Z)$$

for each simple $Z \in \mathcal{C}$ which are linear in the a_i. By solving this system we obtain a basis of the space $\operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}(X,Y)$.

The second possibility is to use the following result.

Lemma 2.8 ([3, Lemma 2.2.])**.** *Let ψ be the spherical structure of $\mathcal{C}$. If $(X,\gamma_X),(Y,\gamma_Y) \in \mathcal{Z}(\mathcal{C})$ then the map $E_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)$ given by*

$$E_{X,Y}(t) = \frac{1}{\dim \mathcal{C}} \sum_{i=1}^{n} \dim X_i \phi_i(t)$$

where $\phi_i(t)$ is given by

$$
\begin{array}{ccccc}
X & \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}(X_i)} & X \otimes (X_i \otimes X_i^*) & \xrightarrow{a^{-1}_{X,X_i,X_i^*}} & (X \otimes X_i) \otimes X_i^* \\
\Big\downarrow {\scriptstyle \phi_i(t)} & & & & \Big\downarrow {\scriptstyle \gamma_X(X_i) \otimes \mathrm{id}_{X_i^*}} \\
 & & & & (X_i \otimes X) \otimes X_i^* \\
 & & & & \Big\downarrow {\scriptstyle \mathrm{id}_{X_i} \otimes t \otimes \mathrm{id}_{X_i^*}} \\
 & & & & (X_i \otimes Y) \otimes X_i^* \\
 & & & & \Big\downarrow {\scriptstyle a_{X_i,Y,X_i^*}} \\
 & & & & X_i \otimes (Y \otimes X_i^*) \\
 & & & & \Big\downarrow {\scriptstyle \psi_{X_i} \otimes \gamma_Y(X_i^*)} \\
Y & \xleftarrow[\mathrm{ev}_{X_i^*} \otimes \mathrm{id}\, Y]{} & (X_i^{**} \otimes X_i^*) \otimes Y & \xleftarrow[a^{-1}_{X_i^{**},X_i^*,Y}]{} & X_i^{**} \otimes (X_i^* \otimes Y)
\end{array}
$$

is a projection from $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ onto $\operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}((X,\gamma_X),(Y,\gamma_Y))$.

Thirdly there are the isomorphisms of the adjunction of the induction functor. Thus by $\operatorname{Hom}_{\mathcal{C}}(V,Y) \cong \operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}(I(V),(Y,\gamma))$ for $X \in \mathcal{C}$ and $(Y,\gamma) \in \mathcal{Z}(\mathcal{C})$ we can compute morphisms out of images of the induction functor. The adjunction isomorphisms are constructive and given by

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(V,Y) &\cong \operatorname{Hom}(I(V),(Y,\gamma)) \\ f &\mapsto \sum_i \phi_i(f)\circ p_i \end{aligned} \tag{2.2}$$

where $p_i\colon \bigoplus_i (X_i\otimes V)\otimes X_i^* \to (X_i\otimes V)\otimes X_i^*$ are the projections and

$$\phi_i(f) = \dim(X_i)\cdot \begin{array}{ccc} (X_i\otimes V)\otimes X_i^* & \xrightarrow{\mathrm{id}_{X_i}\otimes f\otimes \mathrm{id}_{X_i^*}} & (X_i\otimes Y)\otimes X_i^* \\ & & \Big\downarrow{\scriptstyle \gamma(X_i)\otimes \mathrm{id}_{X_i^*}} \\ & & (Y\otimes X_i)\otimes X_i^* \\ & & \Big\downarrow{\scriptstyle a_{Y,X_i,X_i*}} \\ & & Y\otimes (X_i\otimes X_i^*) \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_Y\otimes \psi_{X_i}\otimes \mathrm{id}_{X_i^*}} \\ Y & \xleftarrow[\mathrm{id}_Y\otimes \mathrm{ev}_{X_i^*}]{} & Y\otimes (X_i^{**}\otimes X_i^*) \end{array}$$

The inverse is given by $\operatorname{Hom}(I(V),(Y,\gamma))\to \operatorname{Hom}_{\mathcal{C}}(V,Y)\colon g\mapsto p_0\circ g$. A proof can be found in [3, text]

In practice we will need all three options. Whenever we need to full space of morphisms option one is by far the fastest. But whenever we only need some morphisms - especially between larger objects - it is useful to use option two or three to obtain those. Between them option three is faster. Thus whenever we know that we deal with morphisms out of an objects in the image of I we will proceed with option three, otherwise with option two. That this is practical we will see in the next section.

2.2. Computing Simple Objects in $\mathcal{Z}(\mathcal{C})$.

Lemma 2.9. *Every simple $(Z,\gamma_Z)\in\mathcal{Z}(\mathcal{C})$ arrises as a subobject of $I(X_i)$ for some simple X_i.*

Proof. It exists j such that

$$0\neq \dim \operatorname{Hom}_{\mathcal{C}}(F((Z,\gamma_Z)),X_j) = \dim \operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}((Z,\gamma_Z),I(X_j)).$$

Since (Z,γ_Z) is simple there exists a monomorphism $(Z,\gamma_Z)\hookrightarrow I(X_j)$. □

This is already enough to give a naive algorithm that computes all the simple objects in the center of $\mathcal{C}$.

Algorithm 2.10.
In: Simple objects $X_1,...,X_n$ of a spherical fusion category such that $\dim C\neq 0$.

Out: A complete list of simple objects of $\mathcal{Z}(\mathcal{C})$.

1. Let $X_1, ..., X_n$ be the simple objects of $\mathcal{C}$. Compute all $I(X_i)$.
2. For all i compute the simple subobjects of $I(X_i)$.
3. Reduce the list to non-isomorphic simple objects.

There are two black-boxes in the algorithm: We have to compute subobjects in $\mathcal{Z}(\mathcal{C})$ and we have to check for isomorphy. The latter is straight forward at least for simple objects. If Z_i, Z_j are simple in $\mathcal{Z}(\mathcal{C})$ then since $\mathcal{Z}(\mathcal{C})$ is abelian Schur's lemma allows to conclude from

$$\dim \mathrm{Hom}(Z_i, Z_j) \geq 1 \iff Z_i \simeq Z_j$$

whether Z_1 and Z_2 are isomorphic. On the other hand computing subobjects in a general setting is hard. In the next section we will discuss how this can be done for semisimple categories.

2.3. **How to Compute Subobjects in semisimple Categories.** Let $\mathcal{C}$ be a semisimple category. It is well known, that semisimple abelian categories with finitely many simple objects are (as abelian categories) equivalent to a direct sum of finite dimensional vector space categories. For the precise construction see [6]. Thus if $X_1, ..., X_n$ are the simple objects in $\mathcal{C}$ then

$$\mathcal{C} \simeq \bigoplus_{i=1}^{n} \mathrm{Vec}$$

.

Therefore any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be characterized by a family of vector space morphisms. Moreover we can identify $\bigoplus_{i=1}^{n} \mathrm{Vec}$ with a (not full) subcategory of Vec by taking the direct sum of all objects, respectively morphisms, of the families. Thus after fixing bases for the vector spaces every morphism is encoded by a matrix.

Consider an object $X \in \mathcal{C}$ with $\dim \mathrm{End}(X) = d$. Then $\mathrm{End}(X)$ is a vector space with a basis $\mathrm{id}_X, f_1, ..., f_{d-1}$. Let n be the number of simple objects in the decomposition of X into simple objects. Then after fixing an order of the summands any morphism in $\mathrm{End}(X)$ is given by a $n \times n$ matrix. Let $m_1, ..., m_{d-1}$ be the matrices corresponding to $f_1, ..., f_{d-1}$. Then whenever $d \neq 0$ there exist non-trivial eigenvalues and eigenspaces for the matrices $m_1, ..., m_{d-1}$. Every non-trivial eigenspace $\ker(m_i - \lambda I_n)$ yields a non-trivial subobject $\ker(f_i - \lambda\, \mathrm{id}_X)$. In this manner X can be decomposed inductively.

Remark 2.11. Another viable approach is to compute the endomorphism algebra and computing central primitive idempotents. The big disadvantage here is that the endomorphism space has to be computed in its entirety which is mostly not used in any way afterwards.

2.4. **Refining the Algorithm.** Following algorithm 2.10 will result in redundent computations. We can minimize the nessecary calculations by using some combinatorics of the center construction.

Let $Z \in \mathcal{Z}(\mathcal{C})$ be simple. Then for every simple $X_i \in \mathcal{C}$ we have

$$\dim \operatorname{Hom}_{\mathcal{Z}(\mathcal{C})}(Z, I(X_i)) = \dim \operatorname{Hom}_{\mathcal{C}}(F(Z), X_i)$$

due to the adjunction of F and I. This implies that every simple object of $\mathcal{Z}(\mathcal{C})$ occurs in every induction $I(X_i)$ with multiplicity $\dim \operatorname{Hom}(F(Z), X_i)/\dim \operatorname{End}(Z_i)$. Thus it may allow some speedup by factoring out already known simples from the other inductions.

Example 2.12. Consider the category of finite dimensional S_3-graded $\mathbb{C}$-vectorspaces $\mathrm{Vec}_{S_3}^{\omega}$ twisted by a cocycle ω. From the Grothendieck ring we know that simple objects in $\mathcal{Z}(\mathrm{Vec}_{S_3}^{\omega})$ lie over $\langle (), (12) + (23) + (13), (123) + (132)\rangle$. Indeed if we compute the decomposition of $FI(X_i)$ where $X_i = (), (12), (23), (13), (123), (132)$

$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

we see that there must be two non-isomorphic simple objects over $(12) + (13) + (23)$ and three non-isomorphic simple objects over $(123) + (132)$, otherwise the multiplicities do not match. Also we conclude that $I(12) \simeq I(13) \simeq I(23)$ and $I(123) \simeq I(132)$. Thus we need only to compute $I(())$, $I(12)$ and $I(123)$. By looking at the dimensions of the simple objects in the center and recalling that $\dim \mathcal{Z}(\mathrm{Vec}_{S_3}^{\omega}) = \left(\dim \mathrm{Vec}_{S_3}^{\omega}\right)^2 = 36$ we obtain dimension information for the remaining simple objects:

$$36 - (2 \ \cdot 3^2 + 3 \cdot 2^2) = 6$$

Thus there can be either six simple objects lying over $()$ or two over $()$ and one over $2 \cdot ()$. It becomes clear quickly that there can be only two non-isomorphic half-braidings on $()$.

Thus we refine the algorithm in the following manner.

Algorithm 2.13.
In: Simple objects $X_1, ..., X_n$ of a spherical fusion category such that $\dim C \neq 0$.
Out: A complete list of simple objects of $\mathcal{Z}(\mathcal{C})$.

1. Order the X_i by Frobenius-Perron dimension. We have $X_1 = \mathbb{1}$
2. Compute $Z_1 = I(X_1)$

3. $S := [\,]$
4. Compute simple subobjects of Z_1 and add them to S
5. For all $j = 2, ..., n$
   5.1. Let $S' := \{s \in S \mid \mathrm{Hom}_{\mathcal{C}}(F(s), X_j) \neq 0\}$
   5.2. If $\bigoplus_{s \in S'} s \simeq F(Z_j)$ then break
   5.3. Compute $Z_i = I(X_i)$
   5.4. Iteratively build the quotient by the $Z_i' = Z_i / \bigoplus_{s \in S'} s$
   5.5. compute all simple subobjects of Z_i' and add all new ones to S

Note that the quotients Z_i/s are the cokernels of the embeddings $s \hookrightarrow Z_i$, which are basis vectors of $\mathrm{Hom}(s, Z_i)$.

3. Non-Split Centers

In this section we want to take a look at some properties of the center when the ground field $\Bbbk$ is not algebraically closed. Thus for this chapter if not otherwise stated let $\mathcal{C}$ be a spherical fusion category over $\Bbbk$ such that $\dim \mathcal{C} \neq 0$.

Definition 3.1. Let $\Bbbk \hookrightarrow \mathbb{K}$ be a field extension. We define the category $\mathcal{K} \otimes \mathcal{C}$ to have the same objects as $\mathcal{C}$ and morphism spaces are given by

$$\mathrm{Hom}_{\mathbb{K} \otimes \mathcal{C}}(X, Y) = \mathbb{K} \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

If $\mathcal{C}$ is split semisimple then $\mathbb{K} \otimes \mathcal{C}$ is again a spherical fusion category. If not it might occur that an endomorphism algebra gains idempotents. Consider the example that an endomorphism algebra is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra, i.e has a genarator with minimal polynomial $x^2 + 1$, then tensoring with $\mathbb{C}$ will yield new idempotents corresponding to the the eigenvalues $\pm i$ of the generator. Thus $\mathbb{K} \otimes \mathcal{C}$ is no longer karoubian hence not abelian. This means we will need to consider the karoubian closure $\overline{\mathbb{K} \otimes \mathcal{C}} := \mathrm{Kar}(\mathbb{K} \otimes C)$, which is again a spherical fusion category.

Definition 3.2. We define a functor $\mathbb{K} \otimes - : \mathcal{C} \to \overline{\mathbb{K} \otimes \mathcal{C}}$ mapping objects to themselves and morphisms $f \in \mathcal{C}$ to $1 \otimes f$.

Remark 3.3. By definition 3.2 we will consider $\mathbb{K} \otimes X$ for an object $X \in \mathcal{C}$ as an object of the karoubian closure $\overline{\mathbb{K} \otimes \mathcal{C}}$.

In the algebraically closed case it is well known, that $\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$. This is no longer true when we drop the assumption of $\Bbbk$ beeing algebraically closed. But after some refinement something similar holds.

We will use the slightly altered definition of the Frobenius-Perron dimension of $\mathcal{C}$ introduced in [5].

Definition 3.4. Let $X_1, ..., X_n$ be the simple objects of $\mathcal{C}$.

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{i=1}^{n} \frac{\mathrm{FPdim}(X_i)^2}{\dim_{\mathrm{End}(\mathbb{1})} \mathrm{End}(X_i)}$$

In our case $\dim_{\mathrm{End}(\mathbb{1})} \mathrm{End}(X_i) = \dim_{\Bbbk} \mathrm{End}(X_i)$ since we assume a fusion category with $\mathrm{End}(\mathbb{1}) = \Bbbk$. This seems to be a more precise definition since now we obtain the desired result.

Theorem 3.5 ([5, Chapter 3])**.**

$$\mathrm{FPdim}(\mathcal{Z}(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$$

This immediatly implies nice behaviour of the non-split simple objects after extension of scalars.

Lemma 3.6. *Let $Z \in \mathcal{Z}(\mathcal{C})$ be simple. Then $\overline{\Bbbk} \otimes Z$ decomposes into a direct sum of the form $a \cdot \sum Z_i$ where the Z_i are non-isomorphic and have the same Frobenius-Perron dimension.*

Proof. Let $\overline{\Bbbk} \otimes Z = a_1 Z_1 \oplus \cdots \oplus a_n Z_n$. by the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} a_i \cdot \mathrm{FPdim}(Z_i)\right)^2 \leq \left(\sum_{i=1}^{n} \mathrm{FPdim}(Z_i)^2\right)\left(\sum_{i=1}^{n} a_i^2\right)$$

$$\iff$$

$$\frac{\left(\sum_{i=1}^{n} a_i \cdot \mathrm{FPdim}(Z_i)\right)^2}{\sum_{i=1}^{n} \mathrm{FPdim}(Z_i)^2} \leq \sum_{i=1}^{n} a_i^2. \tag{3.1}$$

Thus theorem 3.5 forces equalities in 3.1 for all simples Z. These equalities hold if and only if $\frac{a_i}{\mathrm{FPdim}(Z_i)} = \frac{a_j}{\mathrm{FPdim}(Z_j)}$ for all i, j.

Now let $A = \mathrm{End}\, Z$ be the endomorphism ring of Z. By assumption A is a simple division algebra. Consider the extension $Z(A) \otimes_{\Bbbk} A$ wich is as $Z(A)$-algebra isomorphic to $\bigoplus_{i=1}^{[Z(A):\Bbbk]} A$. Finally A is central simple as $Z(A)$-algebra and hence has a splitting field $\mathbb{K}$ such that $A \cong \mathrm{Mat}_{n\times n}(\mathbb{K})$. We conclude that $\mathbb{K} \otimes A \cong \bigoplus_{i=1}^{[Z(A):\Bbbk]} \mathrm{Mat}_{n\times n}(\mathbb{K})$ as $\mathbb{K}$-algebras. This implies that $\mathbb{K} \otimes Z$ decomposes into $[\mathbb{K} : \Bbbk]$ non-isomorphic simple objects, each with multiplicity n.

This forces all $a_i = n$ and thus all Frobenius-Perron dimensions to be equal. □

Remark 3.7. If $\Bbbk$ is a finite field the statement above is even stronger since all finite dimensional division algebras over a finite field are fields. Thus a simple object Z decomposes into non-isomorphic simples with multiplicity one.

Example 3.8. the case that a simple object occurs with multiplicity greater than one from lemma 3.6 can occur. Let $\mathbf{Q}$ be the quaternion group with eight elements and consider $\mathrm{Vec}_{\mathbb{Q}}(\mathbf{Q})$ be the category of $\mathbf{Q}$-graded vector spaces over $\mathbb{Q}$. Then there is a central object lying over $4 \cdot \mathbf{1}_{\mathbf{Q}}$ corresponding to the four-dimensional irreducible $\mathbb{Q}$-representation of $\mathbf{Q}$. This object has (similar to the representation) an endomorphism ring isomorphic to the rational quaternions $\mathbb{H}_{\mathbb{Q}}$. Analogously to the representation it decomposes into two copies of the same simple object over $\mathbb{Q}(\sqrt{-1})$.

4. Software framework and implementation

4.1. **Implementation.** Since the linear algebra involved in the algorithm described above it is mandatory to use a computer to apply this algorithm. Thus an interface for explicit fusion categories is needed. We provide such interface as well as the algorithm in a Julia package called `TensorCategories.jl`[2]. The packages allows to construct fusion categories in primarily two ways. Firstly a category can be implemented by defining types for the category, objects and morphisms. Then the nessecary structural methods for direct sum, tensor product, unit, duals, etc. have to be implemented according to the framework. Details on how to do that are to be found in the documentation. The second option is to construct a fusion category from 6j-Symbols.

4.1.1. *Fusion Categories in Julia.* From a theoretical point of view a fusion category is entirely discribed by its $6j$-symbols. But provided only the associators it is not straight forward to explicitely work with the objects and morphisms in this category. Especially if we want to express morphisms in a readable manner, i.e. as families of matrices.

4.2. **The Non-Split Centers of the Ising Category.** We want to examine the center of the Ising category in the case where it is defined over the minimal field of definition.

We recall that the Ising category has three simple objects $\mathbb{1}, \chi, X$ with multiplication $\chi \otimes \chi = 1$, $\chi \otimes X = X \otimes \chi = X$ and $X \otimes X = \mathbb{1} \oplus \chi$. The non-trivial associator isomorphisms are given by

$$\begin{aligned} a_{\chi,X,\chi} &= (-1)\,\mathrm{id}_X \\ a_{X,\mathbb{1},X} &= \mathrm{id}_{\mathbb{1}} \oplus (-1)\,\mathrm{id}_{\chi} \\ a_{X,\chi,X} &= (-1)\,\mathrm{id}_{\mathbb{1}} \oplus \mathrm{id}_{\chi} \\ a_{X,X,X} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathrm{id}_{2X} \end{aligned}$$

Thus we consider it as defined over $\mathbb{Q}(\sqrt{2})$. We start by constructing the Ising category over the desired field.

[2]https://github.com/FabianMaeurer/TensorCategories.jl

```
julia> K,r2 = quadratic_field(2)
(Real quadratic field defined by x^2 - 2, sqrt(2))

julia> I = Ising(K)
Ising fusion category

julia> a,b,c = simples(I)
3-element Vector{SixJObject}:
    𝟙
    χ
    X
```

Then we compute the inductions of the three simple objects and decompose them.

```
julia> ia,ib,ic = induction.([a,b,c])
3-element Vector{CenterObject}:
Central object: 3·𝟙 ⊕ χ
Central object: 𝟙 ⊕ 3·χ
Central object: 4·X

julia> Z = vcat(indecomposable_subobjects.([ia,ib,ic])...)
6-element Vector{CenterObject}:
Central object: 𝟙
Central object: 𝟙
Central object: 𝟙 ⊕ χ
Central object: 𝟙 ⊕ χ
Central object: 2·χ
Central object: 4·X
```

By checking the Hom-spaces it can be determined which simples are non-isomorphic and which simples are not split.

```
julia> [dim(Hom(x,y)) for x in Z, y in Z]
6×6 Matrix{Int64}:
1  0  0  0  0  0
0  1  0  0  0  0
0  0  1  1  0  0
0  0  1  1  0  0
0  0  0  0  2  0
0  0  0  0  0  4
```

We can see that there are five non isomorphic simples of which two are non-split. Indeed we can now examine over which fields they will split. To do so we examine the endomorphism spaces. The object over $2 \cdot \chi$ will split if there is an endomorphism that is a zero-divisor, i.e. if there is a morphism with a non-trivial eigenvalue. Thus we talk a non-trivial endomorphism and consider the splitting field for its minimal polynomial.

```
julia> H = End(C[4])
Vector space of dimension 2 over Real quadratic field defined by x^2 - 2.

julia> minpoly.(basis(H))
2-element Vector{AbstractAlgebra.Generic.Poly{nf_elem}}:
x^2 + 1//4
x - 1
```

So if we extend the field of definition to the splitting field of $x^2 + \frac{1}{4}$, i.e. $\mathbb{Q}(\sqrt{2}, i)$, it will split. We can now see that object splits. Further more also the fifth simple decomposes under this extension.

```
julia> Kx,x = base_ring(I)[:x]
(Univariate polynomial ring in x over real quadratic field defined by x^2 - 2, x)

julia> L,i = NumberField(x^2+1, "i")
(Relative number field of degree 2 over real quadratic field defined by x^2 - 2, i)

julia> indecomposable_subobjects(C[4]⊗L)
2-element Vector{CenterObject}:
Central object: χ
Central object: χ

julia> indecomposable_subobjects(C[5]⊗L)
2-element Vector{CenterObject}:
Central object: 2·X
Central object: 2·X
```

We repreat the process one more time to split also the last two new objects.

```
julia> C2 = C ⊗ L
Drinfeld center of Fusion Category with 3 simple objects

julia> simples(C2)
7-element Vector{CenterObject}:
 Central object: 𝟙
 Central object: 𝟙
 Central object: 𝟙 ⊕ χ
 Central object: χ
 Central object: χ
 Central object: 2⋅X
 Central object: 2⋅X

julia> f,_ = minpoly.(basis(End(C2[6])))
2-element Vector{AbstractAlgebra.Generic.Poly{Hecke.NfRelElem{nf_elem}}}:
 x^2 + 1//4*sqrt(2)*i - 1//4*sqrt(2)
 x - 1

julia> M,a = NumberField(f,"a")
(Relative number field of degree 2 over relative number field, a)

julia> simples(C2 ⊗ M)
9-element Vector{CenterObject}:
 Central object: 𝟙
 Central object: 𝟙
 Central object: 𝟙 ⊕ χ
 Central object: χ
 Central object: χ
 Central object: X
 Central object: X
 Central object: X
 Central object: X

julia> absolute_simple_field(M)[1]
Number field with defining polynomial x^8 + 1//16
over rational field
```

Thus we see that the center of the Ising category splits over the field $\mathbb{Q}(\sqrt[8]{-16})$.

FIGURE 1. The explicit center of Vec_{S_3} where ξ_3 is a third root of unity.

	(12)	(13)	(23)	(123)	(132)
()	1	1	1	1	1
()	−1	−1	−1	1	1
$2 \cdot ()$	$\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$
$(23) \oplus (12) \oplus (13)$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
$(23) \oplus (12) \oplus (13)$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$
$(132) \oplus (123)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	id	id
$(132) \oplus (123)$	$\begin{bmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3^2 \\ \xi_3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3^2 \\ \xi_3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{bmatrix}$
$(132) \oplus (123)$	$\begin{bmatrix} 0 & \xi_3^2 \\ \xi_3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3 \\ \xi_3^2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \xi_3^2 \\ \xi_3 & 0 \end{bmatrix}$

4.3. **Results.** Using the algorithm and especially the software we can compute the center of a given fusion category explicitly by listing the tuples (Z, γ) of simple objects in the center.

4.3.1. *Explicit Centers for Graded Vector Spaces.* Let G be a group and Vec_G the category of finite-dimensional G-graded vector spaces. By theoretical investigations we can characterize the center already. But a unique feature of our approach is the explicit computation of objects together with half-braidings.

Consider $G = S_3$ the symmetric group of order three. We give a full table of objects with half-braidings in figure 1.

4.3.2. *The Fusion Category from the Haagerup Subfactor.* The fusion categories from the Haagerup subfactor are a very important example for fusion categories coming from operator algebras. There are three fusion categories in the equivalence class of the Haagerup fusion category. We consider the one with six simple objects, since it is multiplicity free.

The category $\mathcal{H}$ has the following multiplication table given in figure 4.3.2 while the associativity constraints, i.e. the $6j$-symbols, can be found in [7].

FIGURE 2. Multiplication table of the Haagerup fusion category.

	$\mathbb{1}$	α	α^*	ρ	${}_\alpha\rho$	${}_{\alpha^*}\rho$
$\mathbb{1}$	$\mathbb{1}$	α	α^*	ρ	${}_\alpha\rho$	${}_{\alpha^*}\rho$
α	α	α^*	$\mathbb{1}$	${}_\alpha\rho$	${}_{\alpha^*}\rho$	ρ
α^*	α^*	$\mathbb{1}$	α	${}_{\alpha^*}\rho$	ρ	$\alpha\rho$
ρ	ρ	${}_{\alpha^*}\rho$	${}_\alpha\rho$	$\mathbb{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$
${}_\alpha\rho$	${}_\alpha\rho$	${}_{\alpha^*}\rho$	ρ	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\mathbb{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$
${}_{\alpha^*}\rho$	${}_{\alpha^*}\rho$	ρ	${}_\alpha\rho$	$\alpha^* \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\alpha \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$	$\mathbb{1} \oplus \rho \oplus {}_\alpha\rho \oplus {}_{\alpha^*}\rho$

It is known that the center of $\mathcal{H}$ has twelve objects. We can compute them explicitely such that the forgetful functor is given by the matrix in figure 4.3.2.

4.3.3. *Vercleyen-Singerland Fusion Category.*

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FIGURE 3. The forgetful functor $\mathcal{Z}(\mathcal{H}) \to \mathcal{H}$

$$\begin{matrix} \mathbb{1} \\ \alpha \\ \alpha^* \\ \rho \\ {}_\alpha\rho \\ {}_{\alpha^*}\rho \end{matrix} \quad \begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

FIGURE 4. Multiplication table for $\mathcal{C}$

	e	a	b	aba	t	s	ba	ab
e	e	a	b	aba	t	s	ba	ab
a	a	e	ab	ba	s	t	aba	b
b	b	ba	e	ab	s	t	a	aba
aba	aba	ab	ba	e	s	t	b	a
t	t	s	s	s	$e \oplus ba \oplus ab$	$a \oplus b \oplus aba$	t	t
s	s	t	t	t	$a \oplus b \oplus aba$	$e \oplus ba \oplus ab$	s	s
ba	ba	b	aba	a	t	s	ab	e
ab	ab	aba	a	b	t	s	e	ba

FIGURE 5. The forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$

$$\begin{array}{c} e \\ a \\ b \\ aba \\ t \\ s \\ ba \\ ab \end{array} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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