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Computational aspects of Calogero–Moser spaces

§1. Complex reflection groups

f.d. $\mathbb{C}$-vector space

A complex reflection group (crg) is a finite subgroup $W \subset GL(\mathfrak{h})$ which is generated by reflections, i.e. by elements $s \in W$ s.t. $\operatorname{codim}_{\mathfrak{h}}(\underbrace{\ker(\mathrm{id}_{\mathfrak{h}} - s)}_{\text{fixed space of } s}) = 1$.

A reflection representation of an (abstract) group W is a faithful representation $\rho: W \longrightarrow GL(\mathfrak{h})$ s.t. $\rho(W)$ is a crg.

ρ irred.

The irreducible crg were classified by Shephard–Todd (1954):

* S_n in an irred. refl. rep ("class G_1")
* C_m in a faithful rep ("class G_3")

* $G(m,p,n) :=$ a certain index-p normal subgroup of $S_n \wr C_m$ in the natural rep for $m,n>1$, $p \mid m$, and $(m,p,n) \neq (2,2,2)$. ("class G_2")

  wreath product
* $G_4, \ldots, G_{37}$ ("exceptional groups")

$\rightsquigarrow$ nice book on the classification by Lehrer–Taylor (2009)

My opinion: it is important not to forget about the exceptional groups: are we studying "only" combinatorics or really reflection symmetries in general?

$\rightsquigarrow$ exceptionals often have to be dealt with case-by-case

$\rightsquigarrow$ the computer helps!

$\rightsquigarrow$ Magma, CHAMP, OSCAR

oscar.computeralgebra.de

github.com/ulthiel/champ

In Magma + CHAMP:

```
> ShephardTodd(4);
> ComplexReflectionGroup(4);
```

$\leftarrow$ defined over $\mathbb{Z}[\zeta_3]$ (for ShephardTodd(4))

$\leftarrow$ defined over $\mathbb{Q}(\zeta_3)$ (for ComplexReflectionGroup(4))

which one do you prefer?

§2. Invariants

$\mathfrak{h}$ is a variety with coordinate ring $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*)$.

Let $G \subset GL(\mathfrak{h})$ be finite. The orbit space $\mathfrak{h}/G$ has the structure of an algebraic variety with coordinate ring $\mathbb{C}[\mathfrak{h}]^G$ (<u>invariant ring</u>).

A minimal system of algebra generators of $\mathbb{C}[\mathfrak{h}]^G$ is called a system of <u>fundamental invariants</u>. $\leftarrow$ not unique, but the multiset of degrees is independent of choice.

```
> W:=ShephardTodd(4);
> FundamentalInvariants(W);
> W:=ComplexReflectionGroup(4);
> FundamentalInvariants(W);
```

← defined over $\mathbb{Z}[\zeta_3]$

which one do you prefer?

← defined over $\mathbb{Q}$! (no ζ_3)

⇝ papers by Michel-Marin (2010)

Bourbaki, Chevalley, Serre, Shephard-Todd

<u>Classical fact</u>: $\mathfrak{h}/G$ is smooth $\Leftrightarrow \mathbb{C}[\mathfrak{h}]^G$ is a polynomial ring

i.e. fund. inv. are alg. indep.

$\Leftrightarrow G \curvearrowright \mathfrak{h}$ is a crg

```
> R:=InvariantRing(W);
> Presentation(R);
```

§3. Calogero-Moser spaces

The cotangent bundle of $\mathfrak{h}$ is $T^*\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}^*$.

Physics: $T^*\mathfrak{h}$ is the phase space for the configuration space $\mathfrak{h}$.
A mechanical system is defined by a "Hamiltonian (energy) function" on $T^*\mathfrak{h}$.
Key for deducing the Hamiltonian equations of motion is the symplectic structure on $T^*\mathfrak{h}$: $\omega((y,x),(y',x')) := x'(y) - x(y')$

Now, suppose $\mathfrak{h}$ has symmetries, given by a finite group $\underset{\#}{\overset{1}{G}} \subset GL(\mathfrak{h})$.
$\rightsquigarrow$ induced action $G \curvearrowright T^*\mathfrak{h}$.

could be debated

The phase space taking these symmetries into account should be

$$X_0 := T^*\mathfrak{h}/G = \operatorname{Spec} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^G.$$

Key problem: X_0 is singular ($W \curvearrowright \mathfrak{h} \oplus \mathfrak{h}^* \subset Sp(\mathfrak{h} \oplus \mathfrak{h}^*) \subseteq SL(\mathfrak{h} \oplus \mathfrak{h}^*)$, so no refls)

$\rightsquigarrow$ example of a symplectic singularity ($\rightsquigarrow$ Iain's + Alastair's courses)

```
> WD := SymplecticDoubling(W);
> R := InvariantRing(WD);
> I := Presentation(R);
> X_0 := Scheme(I);
> IsSingular(X_0);
```

Two standard approaches to try to better understand X_0:

$$\text{deformations} \overset{\text{tight connection}}{\longleftrightarrow} \text{resolutions}$$

keep $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ as $\mathbb{C}$-vector space but "deform" the multiplication

$\rightsquigarrow$ Iain + Alastair

Let's focus on deformations.

We don't want random deformations: recall that we have ω on $T^*\mathfrak{h}$, and this descends to a symp. form on X_0^{reg}.

compare: resolution vs. symplectic resolution.

$\rightsquigarrow$ Should only consider deformations deforming ω as well.

Problem: ω lives on X_0^{reg}, not globally on X_0, so cannot directly deform this.

But: ω induces a <u>Poisson bracket</u> globally on $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$ via $\{x_i, x_j\} = 0 = \{y_i, y_j\}$, $\{x_i, y_j\} = \delta_{ij}$.

symp. basis

Lie bracket + Leibniz rule

$\rightsquigarrow$ Want to consider <u>Poisson deformations</u> of X_0.

Suppose from now on that $G = W \subset GL(\mathfrak{h})$ is a crg.

Then all (!) Poisson deformations of X_0 can be constructed as follows.

$=$ group ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$ as $\mathbb{C}$-vs with mult $g \cdot f = {}^{g}f\, g$ ← action of g on f

<u>rational Cherednik algebra</u> (at $t=0$) by Etingof–Ginzburg (2002)

$$\begin{tikzcd}
H_0 := \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W \arrow[rr, squiggly, "\text{deform}"] \arrow[d, dash] & & H_c := \mathbb{C}\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle / \begin{matrix} [x_i, x_j] = 0 = [y_i, y_j] \\ [y_i, x_j] = \sum\limits_{s \in \mathrm{Refl}(W)} (y_i, x_j)_s \cdot c(s) \cdot s \in \mathbb{C}W \end{matrix} \arrow[d, dash] \\
Z_0 := \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W = Z(H_0) \arrow[rr, squiggly] & & Z_c := Z(H_c)
\end{tikzcd}$$

tensor alg.

certain $\in \mathbb{C}$

$c : \mathrm{Refl}(W)/W \longrightarrow \mathbb{C}$

deformation parameters

<u>PBW theorem:</u> $H_c \simeq \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]W$ as $\mathbb{C}$-vs. $\rightsquigarrow$ H_c is a deformation of H_0.

Moreover, $X_c := \operatorname{Spec} Z_c$ is a Poisson deformation of X_0 (<u>Calogero–Moser space</u>).

Recall:

$$H_c := \underset{\text{tensor alg.}}{\mathbb{C}\langle \mathfrak{h} \oplus \mathfrak{h}^* \rangle} \Big/ \begin{array}{l} [x_i, x_j] = 0 = [y_i, y_j] \\ [y_i, x_j] = \sum_{s \in \mathrm{Refl}(W)} (y_i, x_j)_s \cdot c(s) \cdot s \in \mathbb{C}W \end{array}$$

$H_c \overset{\mathbb{C}}{\simeq} \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] W$ means that H_c has a nice basis, namely $\left\{ \underline{x}^\mu \underline{y}^\nu w \;\middle|\; \begin{array}{l} \mu \in \mathbb{N}^n \\ \nu \in \mathbb{N}^n, \end{array} w \in W \right\}$ PBW basis

$\rightsquigarrow$ using the relations, every element (in particular the product of two elements) can be rewritten in PBW form. $\rightsquigarrow$ tell a computer to do this for us!

<u>One important point:</u> we usually don't want to specify c (deformation parameter $c : \mathrm{Refl}(W)/W \to \mathbb{C}$) a priori, rather work <u>generically.</u> $\rightsquigarrow$ Can more generally consider a $\mathbb{C}$-algebra R and $c : \mathrm{Refl}(W)/W \to R$

$\rightsquigarrow H_c$ an R-algebra, $H_c \overset{R}{\simeq} R[\mathfrak{h} \oplus \mathfrak{h}^*] W$

<u>Generic algebra:</u> $\underline{R} := \underset{\text{polynomial ring}}{\mathbb{C}[\varsigma_s \mid s \in \mathrm{Refl}(W)/W]}$, $\underline{c} : \mathrm{Refl}(W)/W \to R,\ s \mapsto \varsigma_s$, $\underline{H} := H_{\underline{c}}$

$\rightsquigarrow$ CHAMP (2015)

```
> H:=RationalCherednikAlgebra(W,0 : Type:="EG");
> H;
> H.1;
> H.6;
> H.1*H.6;
> H.6*H.1;
> eu:=EulerElement(H);
> eu^2;
> IsCentral(eu);
```

§4. Computing X_c (jt. w. Bonnafé)

Consider the generic algebra $\underline{H}$. Let $\underline{Z} := Z(\underline{H})$.

If $h \in \underline{H}$, then $h = \sum_{w \in W} h_w w$ in the PBW basis, $h_w \in \underline{R}[\mathfrak{h} \oplus \mathfrak{h}^*]$.

Consider the map $\mathrm{Trunc}: \underline{H} \longrightarrow \underline{R}[\mathfrak{h} \oplus \mathfrak{h}^*]$, $h \mapsto h_1$.

<u>Lemma</u>: Trunc induces an isomorphism $\underline{Z} \xrightarrow{\sim} \underline{R}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ of (graded) $\underline{R}$-modules.

What we did: found an algorithm to compute $\mathrm{Trunc}^{-1}(f)$ for $f \in \underline{R}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

basically an inductive
deformation procedure

```
> S := W'SymplecticDoublingCoordinateAlgebra;
> f := S.1 * S.3 + S.2 * S.4;
> f^WD.1;
> f^WD.2;
> fH := TruncationInverse(H, f);
> IsCentral(fH);
> fH eq eu;
```

<u>Lemma:</u> If $(f_i)_{i=1,\dots,r}$ is a (minimal) system of algebra generators of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$, then $(\mathrm{Trunc}^{-1}(f_i))_{i=1,\dots,r}$ is a (minimal) system of algebra over R generators of $\underline{Z}$.

Specializing c gives us a (minimal) system of algebra over $\mathbb{C}$ generators of Z_c.

```
> SymplecticDoublingFundamentalInvariants(W);
> CenterGenerators(H : UseDB := false);
```

What we also did: found an algorithm to deform the relations of a presentation of $Z_0 = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ to a presentation of $\underline{Z}$.

Specializing c gives us a presentation of Z_c.

```
> Zpres := CenterPresentation(H : UseDB:=false);
```

What's that all good for?

$\rightsquigarrow$ many explicit results about questions in representation theory and birational geometry.

Here's a fun example.

Calogero-Moser spaces have symplectic singularities. Can we "identify" them?

<u>Theorem:</u> Let $W = G_4$ and $c = (\zeta_3, \zeta_3^2)$. Then the origin of X_c is the unique singular point of X_c and $(X_c, 0)$ is equivalent to $(\overline{\mathcal{O}_{min}}(sl_3(\mathbb{C})), 0)$.

$\uparrow$ can only happen for this c (up to scalars and reordering)

$\uparrow$ closure of the minimal nilpotent orbit.

<u>Proof (!):</u>

```
> zeta := RootOfUnity(3);
```

```
> phi := hom< BaseRing(H) -> CyclotomicField(3) | [zeta, zeta^2] >;
> A8ring := PolynomialRing(Rationals(), 8);
> Zcpres := [ A8ring ! ChangeRing(f, phi) : f in Zpres ];
> A8 := AffineSpace(A8ring);
> Zc := Scheme(A8, Zcpres);
> Zcsing := SingularSubscheme(Zc);
> Zcsing := ReducedSubscheme(Zcsing);
> Zcsing := Scheme(A8, MinimalBasis(Zcsing));
> origin := Zc ! [0,0,0,0,0,0,0,0];
> cone := TangentCone(Zc, origin);
> projcone := Scheme(Proj(A8ring), MinimalBasis(cone));
> IsSingular(projcone);
> Dimension(TangentSpace(Zc, origin));
```

← why?

§5. $X_c^{\mathbb{C}^\times}$ and terminalizations of X_0

$\mathbb{C}\langle \mathfrak{h}\oplus\mathfrak{h}^*\rangle \rtimes W$ has a $\underline{\mathbb{Z}\text{-grading}}$ $(\deg \mathfrak{h}^* = 1, \deg \mathfrak{h} = -1, \deg W = 0)$

$\rightsquigarrow H_c$ has $\mathbb{Z}$-grading $\rightsquigarrow Z_c$ has $\mathbb{Z}$-grading $\rightsquigarrow X_c$ has $\underline{\mathbb{C}^\times\text{-action}}$ (is conical).

One can show: $\mathbb{C}[\mathfrak{h}]^W \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]^W \subset Z_c \rightsquigarrow$ finite $\mathbb{C}^\times$-equiv morphism $\Upsilon_c : X_c \longrightarrow \mathfrak{h}/W \times \mathfrak{h}^*/W$

Only $\mathbb{C}^\times$-fixed point in $\mathfrak{h}/W \times \mathfrak{h}^*/W$ is the origin $\rightsquigarrow X_c^{\mathbb{C}^\times} = \Upsilon_c^{-1}(0)$ (finite set)

Let $\mathcal{C} := \{c : \mathrm{Refl}(W)/W \to \mathbb{R}\}$, $N := \max_{c\in\mathcal{C}} |X_c^{\mathbb{C}^\times}|$, and $\mathcal{C}_{CM} := \{c \in \mathcal{C} \mid |X_c^{\mathbb{C}^\times}| < N\}$

So, $\mathcal{C}\setminus\mathcal{C}_{CM}$ is where $\mathbb{C}^\times$-fixed points are "generic".

Key fact (Bellamy): Let $\pi: Y \longrightarrow X_0$ be a $\mathbb{Q}$-factorial terminalization (exists bei BCHM)
(keep in mind: Y smooth $\Leftrightarrow \pi$ is a crepant ($\Leftrightarrow$ symplectic) resolution!)
Then $\mathcal{C} \simeq \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ naturally as $\mathbb{R}$-vector spaces.

Let $\mathrm{Mov}(\pi)$ be the cone of π-movable line bundles. This decomposes into the ample cones of the various other $\mathbb{Q}$-factorial terminalizations of X_0.
The codim-1 faces of each of the ample cones generate a hyperplane arrangement in $\mathcal{C}$.

Theorem (Bellamy–Schedler–T. + Namikawa): this arrangement $= \mathcal{C}_{CM}$!!

With Bonnafé: explicit algorithm to compute $\mathcal{C}_{CM}$ (based on rep theory of H_c; key is that we can compute generators of Z_c!)

$\leadsto$ allows us to count # of $\mathbb{Q}$-factorial terminalizations of X_0 $\left(\begin{array}{l}\text{\# orbits of chambers}\\ \text{under action of the}\\ \text{Namikawa Weyl group}\end{array}\right)$

```
> H := RationalCherednikAlgebra(W, 0);
> cm := CalogeroMoserFamilies(H : UseDB := false);
> Keys(cm);
```